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THE MATHEMATICS TEACHER

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THE MATHEMATICS TEACHER

Volume XL Number 4

Edited by William David Reeve

Graphical Solution of Equations

By THEODORE R. RUNNING
University of Michigan, Ann Arbor, Mich.

1. Introduction. The problems of our textbooks are stated in words, giving the conditions involved and the quantities to be evaluated. The solution of a problem is readily seen to consist of three stages. The first stage is a translation from one language to another. When we restate a problem in terms of mathematical symbols we are translating, for mathematics is also a language. This translation must be exact, no different shades of meaning are allowed. The second stage consists of the process of solving the equation or equations obtained in the first stage. The third stage is interpreting the values of the unknown quantities obtained in the second stage. The interpretation may be self evident, but cases do arise in which the interpretation is somewhat involved.

2. Basis of Solution. In what follows we shall limit ourselves to a consideration of algebraic equations with real coefficients and one unknown. The determination of complex roots, except in equations of the third and fourth degrees, will not be attempted.

It should be stated at the outset that the roots of an equation will be represented as abscissas of the points of intersection of two curves whose equations are easily obtained from the equation to be solved. The process of graphical solution is the same for all equations and may be illustrated by the solution of the quadratic equation

$$(1) \qquad x^2-2x-3=0,$$

$$(a) \qquad y=x-1,$$

$$(b) \qquad y^2=4.$$

The steps we take in solving this quadratic are the steps we take in solving an equation of any degree. The process is the following. We extract the square root of the left hand member of equation (1), as near as we can, stopping with the second term. This square root is represented by y, equation (a). Equation (1) says that $(x-1)^2=4$. This is expressed by (b). It is evident that the values of x are the abscissas of the points of intersection of the curves (in this case straight lines) $y=x-1$, and $y^2=4$, for if y^2 of (a) be diminished by y^2 of (b) equation (1) is obtained. Equations (a) and (b) are called auxiliary equations and the curves represented by them auxiliary curves.

3. The General Case. We are now ready to state the process of solving an equation of any degree. In order to apply the method described above it is necessary that the equation is of an even degree. If an equation to be solved is of an odd degree it is

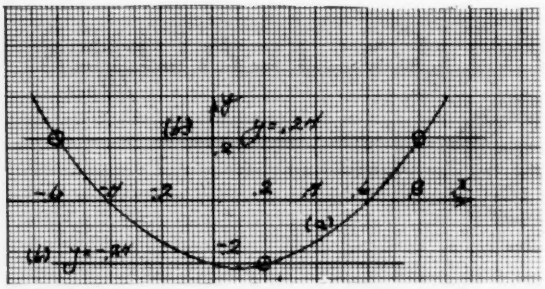


FIG. 1

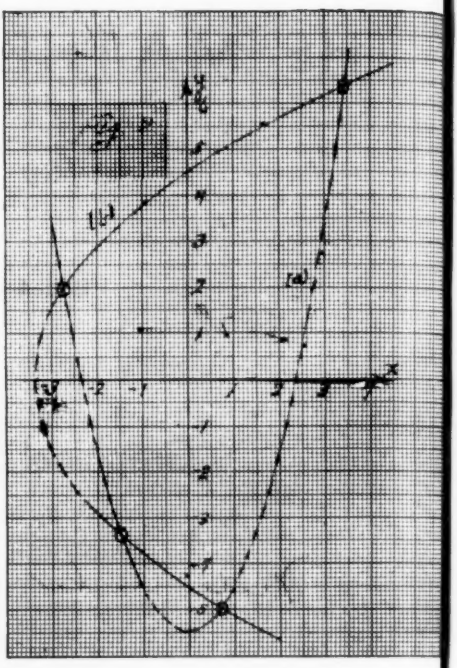


FIG. 2

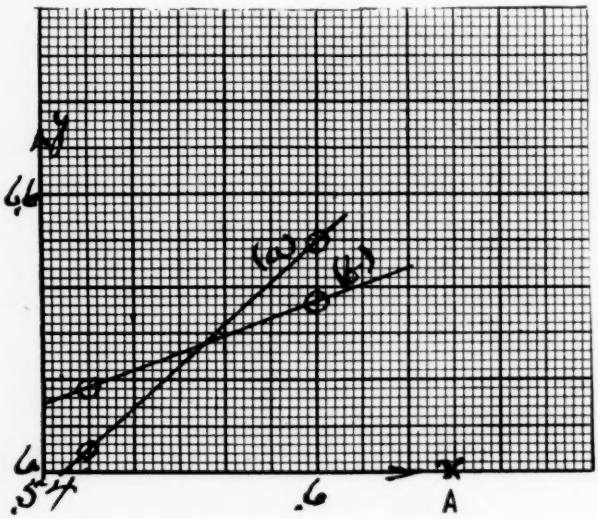


FIG. 3

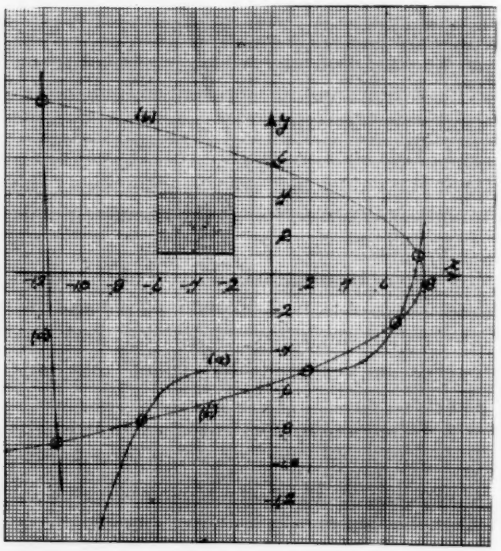


FIG. 5

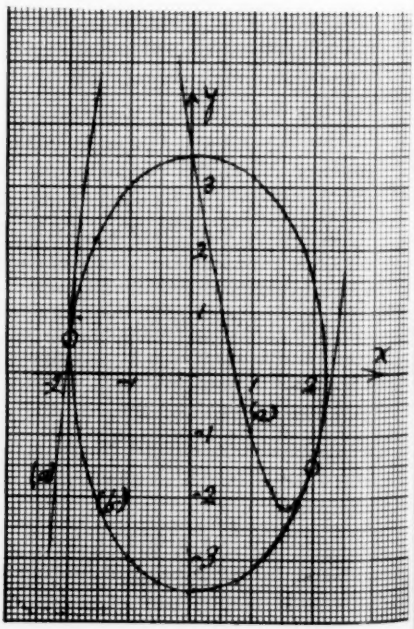


FIG. 4

made even by multiplying both members by the unknown number. This of course introduces the root zero.

Let the general equation be represented by

$$x^{2n}+A_1x^{2n-1}+A_2x^{2n-2}+\cdots$$
$$(A)\qquad +A_nx^n+\cdots+A_{2n-1}x+A_{2n}=0,$$
$$(a)\qquad y=x^n+B_1x^{n-1}+B_2x^{n-2}+B_3x^{n-3}$$
$$+\cdots+B_{n-1}x+B_n,$$
$$(b)\qquad y^2=C_1x^{n-1}+C_2x^{n-2}+C_3x^{n-3}+\cdots$$
$$+C_{-1}-x+C_n.$$

We extract the square root of the left hand member of equation (A) to and including the term A_nx^n. This will be of the form of the right hand member of (a), represented by y. The square of y in (a) diminished by the left hand member of (A) gives the right hand member of (b). This we call y^2.

That the roots of equation (A) are the abscissas of the points of intersection of the auxiliary curves (a) and (b) is seen from the way the auxiliary equations were formed. If we write equation (a) as $y=f(x)$, equation (A) is represented by $[f(x)]^2+k=0$, where k is some function of x. (b) becomes $y^2=[f(x)]^2-[f(x)]^2-k$. y^2 of (a) diminished by y^2 of (b) becomes (A) identically. The number of intersections of the auxiliary curves will be the number of real roots of (A). If the root zero has been introduced it must be discarded. The number of complex roots is evident when the number of real roots is known.

4. Illustrative Examples. Let it be required to solve

$$(2)\qquad x^3-.4x^2-.44x+.096=0.$$

Multiplying the equation through by x we obtain

$$x^4-.4x^3-.44x^2+.096x=0,$$
$$(a)\qquad y=x^2-.2x-.24,$$
$$(b)\qquad y^2=.0576.$$

The auxiliary curves are drawn in Fig. 1. The roots are read as abscissas of the points of intersection. They are $-.6$, $.2$, and $.8$. Zero is an introduced root.

Example 3. Let it be required to solve

$$x^4-11x^2-6x+10=0,$$
$$(a)\qquad y=x^2-5.5,$$
$$(b)\qquad y^2=6x+20.25.$$

We draw rough sketches of the auxiliary curves to locate approximately the location of the points of intersection. The two curves are then drawn with considerable accuracy in the neighborhood of the points of crossing. Fig. 2 is the completed drawing. The roots can be read approximately to the second decimal place. They are -2.73, -1.45, $+.73$, $+3.45$. Interpolations give more extended values, -2.73205, -1.44949, $+.73205$, $+3.44949$

Example 4. Let us try to solve

$$x^4-2x^3+3x^2-4x+5=0,$$
$$(a)\qquad y=x^2-x+1,$$
$$(b)\qquad y^2=2x-4.$$

Upon drawing the auxiliary curves it is seen that they do not intersect. The roots are complex. To obtain the roots we factor the left hand member into two quadratic factors $(x^2+Ax+B)(x^2+Cx+D)$. The equation then becomes

$$(4a)\qquad x^4+(A+C)x^3+(B+D+AC)x^2$$
$$+(AD+BC)x+BD=0.$$

Equating coefficients between (4) and (4a) the following four equations result.

$$A+C=-2,\qquad AD+BC=-4,$$
$$B+D+AC=3,\qquad BD=5.$$

By eliminating B, C, and D from these four equations we obtain a sextic in A. $B=(4+3A+2A^2+A^3)/(2+2A)$. The sextic is

$$A^6+6A^5+18A^4+32A^3$$
$$+21A^2-6A-12=0,$$
$$(a)\qquad y=A^3+3A^2+4.5A+2.5,$$
$$(b)\qquad y^2=14.25A^2+28.5A+18.25$$
$$=14.25(A+1)^2+4.$$

A rough sketch of the auxiliary curves locates the two points of intersection. Only

one need be considered, for the values of A and C are connected by the equation $A+C=-2$. Fig. 3 gives the value of A as .576. Interpolation gives $A=.57564$, $C=-2.57563$, and, using the equation for B in terms of A, $B=2.08816$, $D=2.39445$. The equation to be solved becomes $(x^2+.57563x+2.08816)(x^2-2.57563x+2.39445)=0$. From this the complex roots are easily found. To find the complex roots of a cubic when they occur we may proceed as in the above quartic.

5. Multiple Roots. Let it be required to solve the equation

(5) $x^5-2x^4-8x^3+16x^2+16x-32=0.$

Multiplying by x

$$x^6-2x^5-8x^4+16x^3+16x^2-32x=0,$$

(a) $y=x^3-x^2-4.5x+3.5,$

(b) $y^2=-2.75x^2+.5x+12.25.$

The auxiliary curves are drawn in Fig. 4. It is at once evident that they have some unusual peculiarities. Apparently they are tangent to each other at $(-2, .5)$. It is easily shown that the two curves pass through $(-2, .5)$ with the same slope. Apparently they cross at $(2, -1.5)$. So they do and with the same slope. When two curves are tangent to each other it indicates that two points of crossing have coalesced at the point of tangency. Therefore 2 is a double root. In plotting the curves it was found that the point $(2, -1.5)$ was common to both curves. It was also found that the curves have the same slope at that point. When two curves cross with the same slope it indicates that three points of crossing have coalesced. Therefore 2 is a triple root of equation (5). The equation may be written $(x+2)^2(x-2)^3=0$. The auxiliary curve cross at $x=0$ but zero is an introduced root.

6. Equations of Higher Degree. As illustrations an equation of the eighth degree and one of the twentieth are solved.

Let it be required to solve the equation of the eighth degree

(6) $x^8-10x^6+33x^4-40x^2-x+14=0,$

(a) $y=x^4-5x^2+4,$

(b) $y^2=x+2.$

Make a sketch of the auxiliary curves. After the points of intersection have been located approximately draw the curves with care in the neighborhood of the intersections. The roots of the equation can then be read as abscissas of the points of intersection to the second decimal place. They are the following: -2.00, -1.99, -1.16, -0.82, $+0.73$, $+1.36$, $+1.75$, $+2.14$.

For the last illustration let us find all the real roots of the equation

(7) $x^{20}+4x^{15}+3x^{10}-2x^5+.4x-.07=0,$

(a) $y=x^{10}+2x^5-0.5,$

(b) $y^2=-.4x+0.32.$

From Fig. 5 we read the roots to two decimal places. They are -1.21, -1.12, -0.68, 0.18, 0.66, and 0.77. The equation has only six real roots, it has seven pairs of complex roots. The real roots to five places are -1.20568, -1.12340, -0.67982, $+0.17552$, $+0.65796$, and $+0.76919$.

Something For Council Members to Emulate!

Miss Helen Young, of Winthrop College, Rock Hill, South Carolina, has just sent in twenty-eight subscriptions to THE MATHEMATICS TEACHER. This means twenty-eight active young members of The National Council. If we could have this kind of support from the mathematics students of each college throughout the country, we would soon achieve our goal of ten thousand members. Hats off to Miss Young!—EDITOR

Dynamic Geometry

By JOHN F. SCHACHT, *Bexley High School, Bexley, Ohio*

AND

JOHN J. KINSELLA, *University School, Ohio State University, Columbus, Ohio*

IN 1938 Lloyd wrote: "Traditionally, geometry has been studied in the tenth grade as a system of rigid, non-flexible elements. The author advances the thesis that all geometric diagrams should be assumed as flexible linkages. . . ." The writer felt that the use of motion made the geometric materials more lifelike and, hence, more interesting. He also believed that many geometric properties were elucidated more clearly by the flexible models than by similar static ones. Finally, he pointed out that the variance and invariance of certain geometric elements were more easily demonstrated by jointed rather than fixed devices.

A reading of the literature on mathematical education will show that the use of motion was fairly common in the study of loci. However, too frequently the locus was considered only as a class of points, not the path of a moving point. The difference here is the difference between looking at "slides" and watching a "movie." Another use of motion was in the study of such continuities as that of angle measurement. If the possibility of zero and negative arcs is admitted, then any angle formed by two lines touching or intersecting a circle can be measured by one-half the sum of the intercepted arcs. The possibility of revealing many other continuities by the use of dynamic models is one of the strongest arguments for the use of them. Incidentally, the contributions of these devices to seeing things as a whole in all their relationships puts them in step with some of the latest emphases in educational psychology. Finally, an increased use of models would make the learning of slow-learners easier, if studies of the characteristics of these learners have any validity.

The remainder of this article will be devoted to these purposes, (1) to describe a few of these dynamic models and some of the large number of theorems revealed by them, (2) to sketch briefly classroom procedures in which the use of these flexible devices is an integral part of the learning process, (3) to provide a few sample guide sheets for illustration and, finally, (4) to discuss the making of the devices themselves.

THE DEVICES

The Dynamic Triangle Device (See Figure 1.)

The triangle is made of a durable material attractively colored. The sides are constructed so that a great variety of sizes and shapes of triangles can be easily obtained. They are scaled for direct reading of the length of the sides. Protractors are mounted rigidly at the vertices and on three movable points on the sides. Auxiliary, colored, elastic threads can be hooked to the vertices and movable points.

Many properties and relationships involving angles, sides, angle bisectors, medians, altitudes and segments terminated by the sides can be discovered and verified. By using two or more triangles patterns of congruence and similarity can be studied.

The Dynamic Quadrilateral Device (See Figure 2.)

The sides of the quadrilateral are extensible and the midpoints of the sides are maintained mechanically. Scaled sides and mounted protractors permit the measurement of sides and angles by direct reading. Elastic threads can be attached to any combination of vertices and midpoints.

A multitude of properties and relationships of the variable parts of all types of quadrilaterals can be illustrated and checked with this device.

The Dynamic Circle Device (See Figure 3.)

This device consists of a large circular protractor and three linearly scaled members having a common pivot. The protractor is ruled with radial lines and equally spaced concentric circles. The common pivot can be placed within, on or outside the circle. The scaling makes easy the reading of tangent segments and distances of chords from the center.

The device is admirably suited for the study of the following:

1. Angle measurement
2. The relation of chords and arcs.
3. The relation of chords and their distance from the center.
4. Relations between tangents, secants and chords.
5. Relations between diameters and chords.
6. Certain loci.
7. Relations between the sides of a right triangle and the altitude on the hypotenuse of a right triangle.

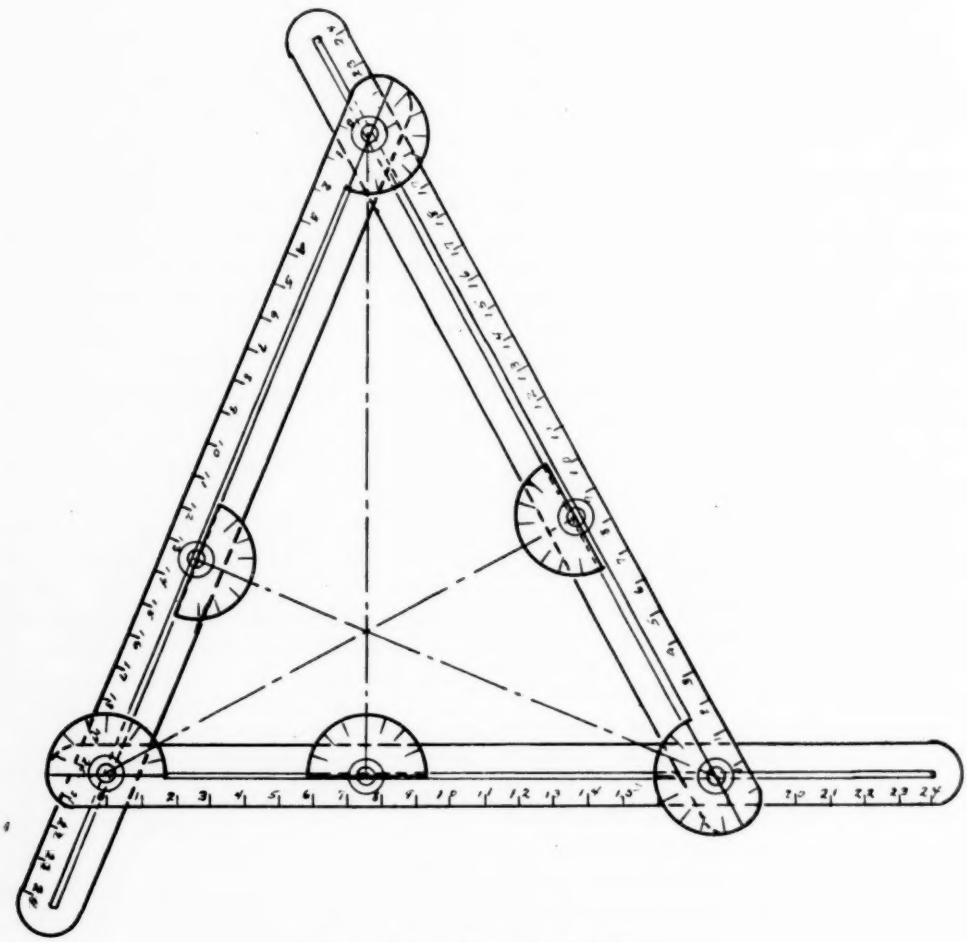


FIG. 1. Dynamic Triangle Device.

Other Devices

Among other useful devices are the following:

1. A triangle device with midpoints maintained mechanically for a study of median relationships, and those involving the segment joining the midpoints of two sides of a triangle.
2. A device to show how diagonal relationships determine types of quadrilaterals.

3. A quadrilateral inscribed in a circle.
4. Basic and special loci. (The locus in each has a characteristic color which facilitates the study of the possible combinations of simple loci.)

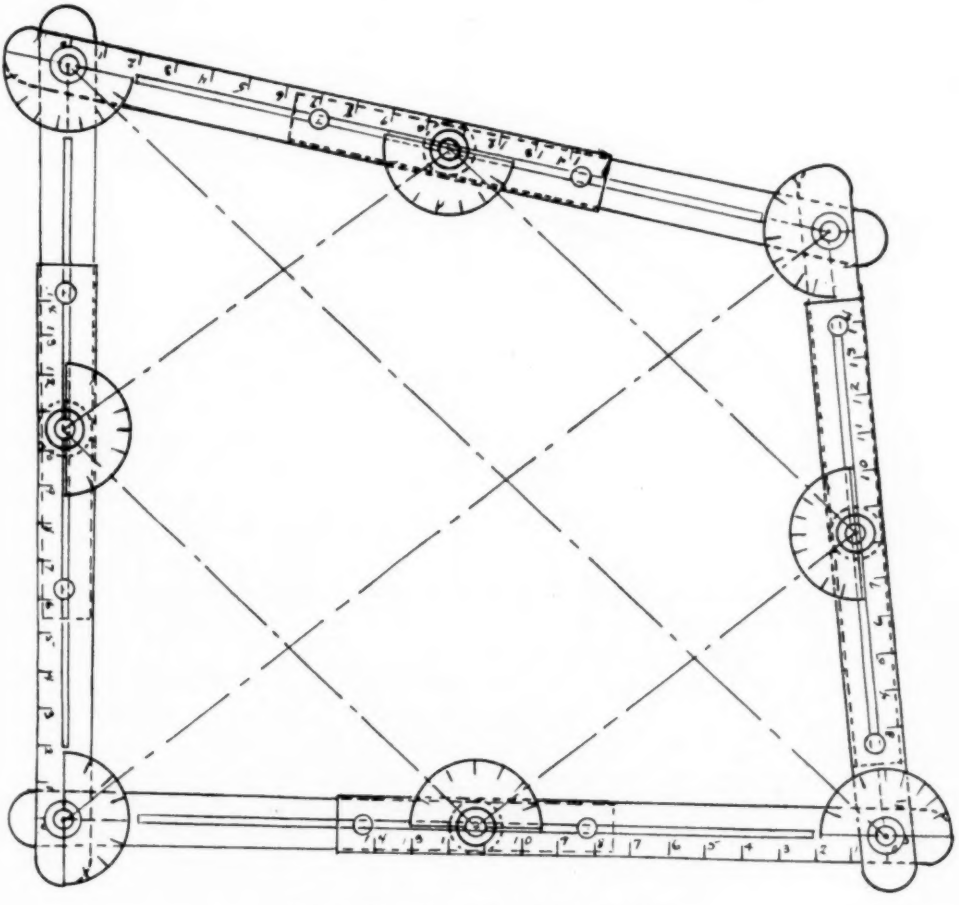


FIG. 2. Dynamic Quadrilateral Device.

CLASSROOM PROCEDURES

1. It is essential that all students have the opportunity to participate in experiments individually or in small groups. This necessitates sufficient duplication of the various devices so that at certain times all students may work as a team under teacher guidance.
2. The groups should be formed as democratically as possible so that each person derives maximum benefit from his participation. Special interests and aptitude are factors in group-forming.
3. After the devices have been distributed, opportunities for manipulating them should be provided. This preliminary exploration should lead to the statement of various problems or hypotheses.
4. These proposals are listed. Group discussion leads to the elimination of some of these. Those remaining are ready to be checked by measurement.
5. The measurements are recorded in a table. Some of the data are found to be irrelevant. Some of the relationships are found to be true only in special cases, i.e., these data are not representative. The approximateness of measurement has to be taken into account in judging the correctness of the relationship.
6. The relationships remaining are first stated in terms of the lettering of a specific diagram. These are called *observations*, which are finally expressed in words. These sentences are the *hypotheses* to be tested.
7. Under the guidance of the teacher the empirical proof of some of these statements is accepted. For the more important and far-reaching ones deductive proofs are attempted in the customary manner. Thus,

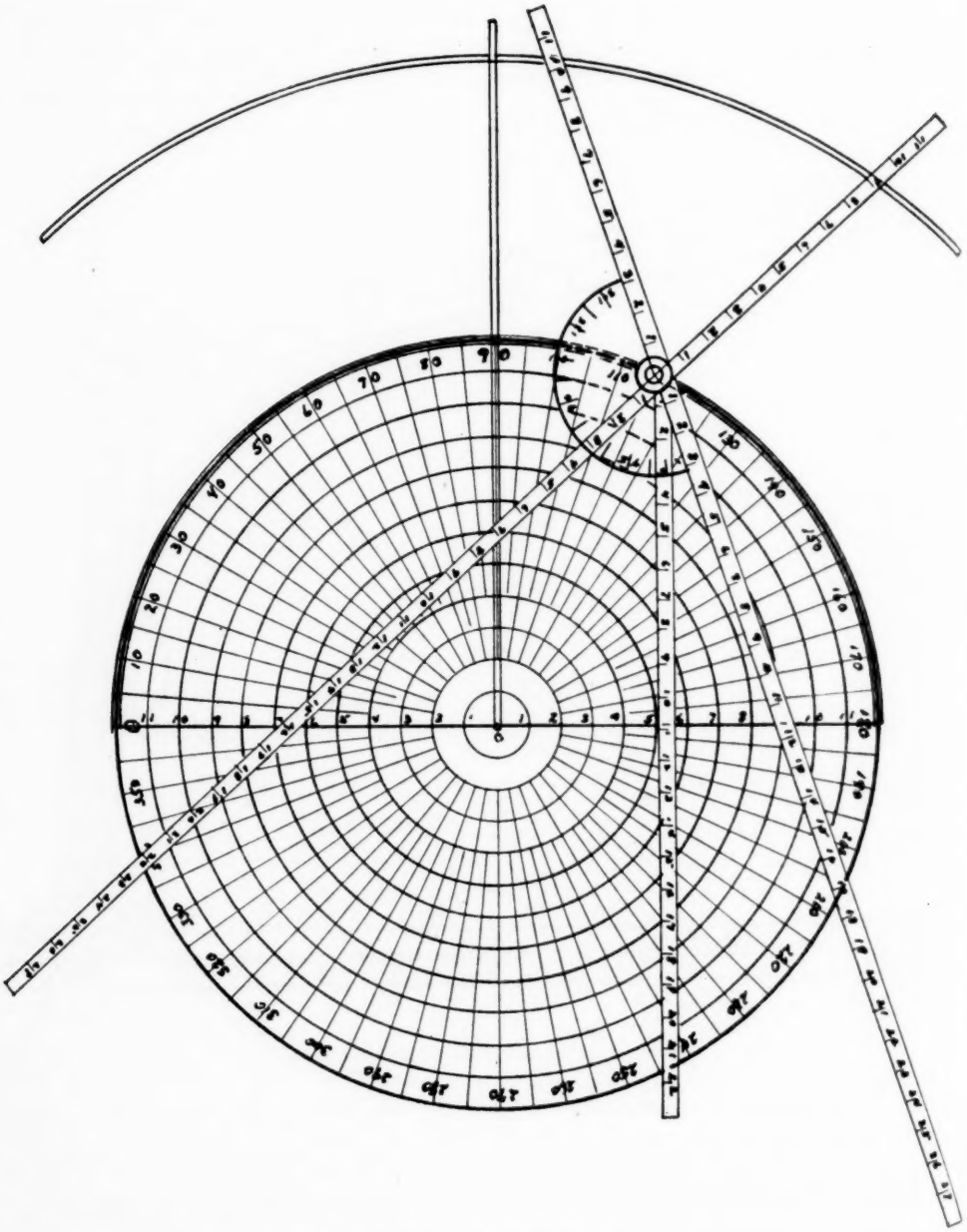


FIG. 3. Dynamic Circle Device.

ample opportunity is provided for acquiring knowledge of geometrical relations through direct experience and discovery, rather than, second-hand, through memorization of some others' statements. At the same time, the meaning of deductive proof is acquired. The notion of sequence is a natural outgrowth of trying to prove statements which require previous proofs for their validity.

8. The proofs are recorded in a notebook for future reference.

It should be noted that these procedures follow very closely the problem solving process as described by Dewey and others. The problems are defined by the students as far as possible, data are gathered and

studied for possible relations, hypotheses are set up and proved or disproved by deduction. At the proper time these stages are abstracted and applied to non-mathematical problems as well.

Use of such procedures will usually bring about noticeable changes in student attitudes. Instead of the passiveness, which is so often the result of reciting an author's outlined proof or of fumbling with an "original" that is original only for the originator, instances of initiative and self-direction become more common. The proposing of problems by the students themselves will probably become a more frequent occurrence. Of course, the students will occasionally have to say "we can't prove that yet" or "we will have to prove this other proposition first." These experiences will be all to the good, however, for the presence of these logical gaps reveal the necessity for sequence and the nature of "if . . . then . . . " type of thinking.

Guide Sheets

When preparing guide sheets, consideration must be given not only to the extent of the student's experience with the investigational technique but also to his ability to work things out for himself. Early in the course considerable guidance will be necessary on all levels of intelligence. Later, as interest and initiative become more evident, the opportunity and privilege of manipulating the device, along with a few oral suggestions, will probably suffice. At any rate, the problems can be of a more general nature and the suggestions less detailed than is advisable earlier.

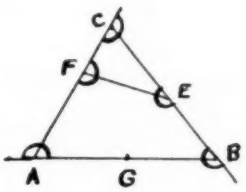
The subject matter of the guide sheets included in this article is developed in the later stages of a geometry course. It is presumed that by this time the student has had considerable experience in the geometry laboratory. For some students there is more guidance than necessary; for others the problems may be too complex. Teachers should prepare guide sheets in such a manner that the purpose of the experiences and the students' maturity are both taken into account.

Sample Guide Sheet 1

Problem

To discover experimentally any relations among angles, segments and sides of a triangle when a line segment is terminated by two sides of a triangle.

The Device (See Fig. 1)



Introduction

Equip the basic triangle device of figure 1 with an auxiliary, elastic segment attached to the movable points, F and E, as indicated.

You may reasily recognize the following possibilities, suggested by the problem:

Case 1. FE may be parallel to AB.

Case 2. F and E may divide AC and BC into proportional segments.

Case 3. Angle BAF may be equal to Angle CEF.

Case 4. F and E may be the midpoints of AC and BC respectively.

Case 5. F may be the midpoint of AC and FE parallel to AB.

Case 6. F and E may be placed in any position whatsoever.

Procedure—for case 1:

Vary the length of either AC or BC. Vary AB. Vary the lengths of all three sides. Note the measurements of all the variables; i.e., the angles, sides and segments. Be sure that, for all positions of the controlled elements, FE is parallel to AB.

Data Table

Trial	$\angle BAF$	$\angle EFC$	$\angle EBA$	$\angle CEF$	$\angle C$	FE	AB	AC	BC	AF	FC	BE	EC
1													
2													
3													
4													
5													

Observations (In terms of the diagram)

1.
2.
3.

List any data you find to be irrelevant.

Hypotheses (Observations generalized)

1.
2.
3.

In a manner similar to the procedure for case 1, investigate cases 2, 3, 4, 5 and 6. Organize your data in tabular form. Write the hypotheses which seem reasonable and acceptable.

Write a deductive proof for cases 4 and 5.

Problems for Further Investigation

1. Consider the relationships that might exist between triangles ABC and FEC.

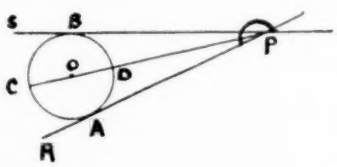
2. Arrange an elastic thread so that it joins the midpoints of all three sides of the triangle ABC. What new relationships seem likely?

Sample Guide Sheet 2

Problem

To investigate any relations there might be among angles, segments and arcs when tangents to a circle and a secant through its center are drawn from a point external to the circle.

The Device (See Figure 3)



Introduction

An analysis of the problem suggests relationships between the following combinations of variables:

Case 1. The tangent segments, AP and BP.

Case 2. The angle formed by the tangents, and the intercepted arcs.

Case 3. The two angles formed by the secant and tangents.

Case 4. A tangent segment and a secant segment or combination of segments.

Case 5. The angle formed by a tangent and the secant and the intercepted arcs.

Procedure

Arrange the parts of the device so that the pivotal point, P, is external to the circle, PS and PR are tangent to the circle at B and A respectively, and PC passes through O, intersecting the circle at D and C.

Change the position of P relative to the circle and note the measurements of the variables, i.e., arcs, angles and segments.

Data Table

(Spaces for any additional data.)

Trial	AP	BP	$\angle BPA$	$\overset{\frown}{BCA}$	$\overset{\frown}{ADB}$									

Observations

1.
2.
3.
4.
5.

Hypotheses

1.
2.
3.

Write deductive proofs for Cases 1, 2 and 3.

Questions for further study

1. What is the locus of P when the length of a tangent segment is constant?

2. Check the validity of the hypothesis, "if the angles formed by two lines with a secant through the center of a circle from an external point are equal, the lines are tangent to the circle."

3. When PC does not pass through 0, how is angle BPC measured?

4. For any secant PC, what is the relation, if any, between BP, PD and PC?

Procurement of the Devices

A resourceful teacher finds many opportunities to provide useful equipment, such as films, slides, construction materials, models, and instruments for class and field work. He does not overlook the potential ability of the student in this respect. A student who makes a model has a rich and lasting experience and feels great satisfaction in his contribution to the geometry laboratory.

However, to construct all the apparatus essential to a well equipped laboratory necessitates, on the part of both the teacher and the students, an expenditure of time out of proportion to the benefits received. The science laboratory is amply equipped. Schools do buy equipment for field mathematics. If durable, dynamic models can make learning more efficient for more students, they, too, should be provided. There is some probability that in the near future these flexible, geometric models, equipped with protractors and scaled segments and arcs for direct reading, will be available commercially.

Bibliography

Lloyd, Daniel B. The Mathematics Teacher, Volume 32, November 1939, "The Teaching of Flexible Geometry," pp. 321–324.

National Council of Teachers of Mathematics, 18th Yearbook, "Multi-sensory Aids in the Teaching of Mathematics," 1945.

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Variation Applied to Problem-Solving*

By BARNETT RICH
High School of Music and Art, New York City

I WOULD like to consider with you the topic "The Application of the Three Types of Variation; Joint, Direct, and Inverse Variation, to the Solution of Typical Problems." By typical problems, I mean more than the typical variation problem where the word variation is explicitly stated. I have in mind such ordinary and common types as motion, investment, area, volume, solution and mixture problems. Because of the frequency and importance of these types, the wide significance of the topic can be readily understood.

How is the topic of variation now being treated in our textbooks and classrooms? In answer, I quote from an article written by N. J. Lennes, "The Function Concept in Elementary Algebra." (Page 59 of the Seventh Yearbook of the National Council of Teachers of Mathematics.) He states,

> The result we get from a study of these texts is about as follows: In the majority of classes in elementary algebra, possibly sixty-five per cent of them or more, the ideas of function and of variation are introduced in a perfunctory way which cannot leave any lasting impression on the minds of the students. In another group, possibly twenty-five per cent, there is a more intensive study of variation at the end of the year. In no case, can functionality or variation be more than one of a large number of topics studied in the course. In the majority of cases, these topics have no organic relation with the rest of the work and appear to have been brought in for extraneous reasons. What these extraneous reasons may be, we can only conjecture. In writing textbooks, it is sometimes thought best to put in a little of everything so that all wants may be satisfied. Certain it is that in the majority of our classes in algebra, not only have we failed to make the function concept an all-pervading or unifying principle, but the consideration of it is so perfunctory that little would be lost if it were left out entirely. The lip service paid to it on a few isolated pages is of very little consequence.
>
> No doubt a small number of teachers are developing the idea of functionality in a sane way, making it an organic part of the course.

Although the article was written a few years ago, can it be said that any real improvement has been made since then?

In the last sentence, Lennes has indicated a major mathematical objective. How can we develop the idea of functionality in a sane way, making it an organic part of the course? It is my contention that the main difficulties that arise in this connection are attributable to the failure of teachers to appreciate the values of applying variation; and furthermore, that teachers are not aware of methods that may be used in the teaching of variation in order that the students may secure these values to an appreciable extent.

Therefore, I intend to consider first, the values to be gained in the application of variation to problem-solving; and second, the methods to be employed in the teaching of the principles of variation so that these values may be realized in the classroom.

First, let us consider the values to be derived. It is in this connection that my recent military experience comes readily to mind. While I was an instructor at the Army Air Forces Navigation Instructors School, I taught and prepared lectures on many scientific subjects related to the navigation of a plane. In numerous situations the principles and techniques of variations proved invaluable in providing the simplest method of explanation and solution.

To demonstrate the power of variation in simplifying the solution of problems, let us work out two problems in inverse variation that were given on one of the New York State Regents Examinations in Intermediate Algebra. In Part III, there appears the following problem:

How many quarts of water must be added to 12 quarts of a mixture that is 15% alcohol to make a mixture that is 10% alcohol?

* Read at a meeting of the Association of Mathematics Teachers of New York City on February 16, 1946.

The data may be represented in the following tabular form:

	Rate or % of alcohol		No. of qt. in solution		Amount or no. of qt. of alcohol	
Original Solution	15%	3	12	2	remains	1
New Mixture	10%	2	M	3	same	1

FORMULA: $RS = A$

Note how all the ratios are indicated in the corners of the respective columns. Since the amount of alcohol remains the same, the R-ratio and the S-ratio are inverse to each other, and the resulting S-ratio equals $\frac{2}{3}$. Two solutions are shown:

Solution I

$\frac{M}{12} = \frac{3}{2}$

$\therefore M = 18$

Solution II

Since M and 12 are in the ratio of 3:2, M may be represented by $3x$, and 12 by $2x$.

If $12 = 2x$, then $x = 6$.

But x or 6 qt. is the amount of water added.

Note how the second solution avoids the use of any fractional equation. Furthermore, in the process of taking a ratio, the decimals were eliminated, and the numbers 15 and 10 were reduced to a ratio of 3:2.

We are now ready to carry the process a step forward and realize an important advantage to be gained once the student has understood this method of solution. By keeping the ratios constant, students can now create any number of new problems which will arrive at exactly the same result. Keeping the rate-ratio at 3:2, the actual per cents might have been 15.6% and 10.4%, or 3.33% and 2.22%, or 36% and 24%. Furthermore, the process of arriving at the result remains exactly the same. The importance of this procedure is the resulting gain on the part of the student in understanding and creative problem formulation rather than manipulation of algebraic equations.

On the same examination, there occurs the following motion problem:

A boy walked out into the country at the rate of 4 miles per hour and returned over the same road at the rate of 3 miles per hour. If the entire trip took $3\frac{1}{2}$ hours, how far into the country did he walk?

Following is the tabular representation of the data of the problem:

	Rate in MPH		Time in hr.		Distance in mi.	
Trip going	4	4	$3x$	3	remains	1
Trip returning	3	3	$4x$	4	$(=12x)$ same	1

FORMULA: $RT = D$

$7x = 7/2$

$x = \frac{1}{2}$

Therefore, the distance $12x = 6$ miles.

As in the case of the alcohol example, we can now secure variety in this one problem by changing the respective rates without changing the rate-ratio, and, in this manner, retain our solution for every new problem created.

We can now bring out an important, if not the most important, value of all. The numerical data and the algebraic work in the above problem are not dependent on the type of problem used, in this case, a motion problem. In fact, it might have been any example of the $xy = z$ variety. To illustrate, the problem might have concerned two rectangles of equal areas whose lengths are respectively 4 feet and 3 feet. If the sum of their widths is $3\frac{1}{2}$ feet, what is the area of each? The formula, $LW = A$, being of the $xy = z$ character, leads to identically the same result except for the units involved. If the headings of the table were replaced with length, width and area of

rectangle and the trips by rectangles I and II, no further change would be required to complete this new problem.

Let us now return to the first solution and see how power can be developed and understanding broadened by utilizing one set of data in numerous applications of the same related type. Note the scientific possibilities.

$WD_f = M$
(for balanced lever)

Weight (lb.)	Distance from fulcrum (in.)	Moment (lb. in.)
3	2	1
2	3	1
15	12	remains
10	x	same

$PN = C$

Price per lb. (¢)	Number of lb.	Cost (¢)
3	2	1
2	3	1
15	12	remains
10	x	same

$PV = k$
(for an enclosed gas)

Pressure per cu. in. (lb.)	Volume (cu. in.)	Constant
3	2	1
2	3	1
15	12	remains
10	x	same

$sn = p$
(for an equilateral polygon)

Length of side (in.)	Number of sides	Perimeter (in.)
3	2	1
2	3	1
15	12	remains
10	x	same

To the four illustrated variations of the original mixture problem could be added an almost endless number of related problems, all of the $xy = z$ or factor × factor = product type. For example:

Mathematical Formulas	Scientific Formulas
fraction × denominator = numerator	density × volume = mass
principal × rate of interest = annual interest	height × density = pressure
base × rate = percentage	pressure × area = force
list price × rate of discount = discount	force × distance = work
assessed value × tax rate = tax	mass × velocity = momentum
	efficiency × input = useful output

I have found it very useful when dealing with this topic of variation in problem-solving, to have at hand cards that contain the headings that may be used in connection with the formulas mentioned above and others of a similar character. By means of this simple expedient, once a problem of the $xy = z$ type has been worked out, the heading may be replaced by another card with the heading for another problem of the same type. Thus, as has been shown above, a mixture problem by this simple transformation can become a balanced lever problem; or an enclosed gas problem; or one concerned with price, number and cost; or a scientific problem involving wave length, frequency and velocity; and so on.

Thus, a mathematics teacher, or a science teacher, can become a quick change artist and create a new problem as fast as he can change a heading. In fact, any teacher of any subject who is concerned with formulas, where the elements involved are multiplied or divided, can make use of the procedures that have been shown.

However, it is not sufficient that our students realize the exceedingly close relationship that exists between problems by virtue of the similarity that exists in their formula structure. Once the conditions of a problem have been tabulated, it is not a difficult matter for students *to create new*

problems from the original as long as the new problem possesses this similarity. Developing this creative ability on the part of a student has unusual possibilities for growth in understanding, power and satisfaction, and for integration and correlation of subject matter.

To illustrate this point of creative ability, let us replace the mixture problem by an interest problem by changing the heading. We might vary the data a little to make it more appropriate by adding three zeros to the original 12.

Our new table will then appear as follows:

	Rate of interest	Principal	Annual interest
	3 / 2	2 / 3	1 / 1
Original investment	15%	12000	remains
New investment	10%	x	same

FORMULA: $RP = I$

Now, let the student make up the investment problem which might yield the table above. He might suggest that the problem concerns a stockholder whose rate of profit on an investment of \$12,000 has diminished from 15% to 10%. He, the stockholder, would like to know how much must be added to his original investment in order to obtain the same income. The solution is obviously \$6000. Nothing need be changed once the original problem has been solved except for the addition of three zeros to the original answer. It is astonishing to know to what extent pupils can think up new problems for given data. You will discover that they can think of a great many that are in your texts as well as a great many more that are not.

To illustrate the possibilities of the application of variation to problem-solving, the following motion problem shows what can be done when different measurements are used.

An aviator travels from his base at the rate of 350 feet per second over a course of 1000 miles. He returns at the rate of 250 feet per second over the same course. The return trip occupied 20 minutes longer than the outgoing one. What was the total time consumed for the trip?

The conditions of the problem may be tabulated as follows:

	Rate (ft. per sec.)	Time (min.)	Distance (miles)
	7 / 5	5 / 7	1 / 1
Trip going	350	$5x$	remains (1000)
Trip returning	250	$7x$	the same

Solution. $7x - 5x = 20$

$2x = 20$ or $x = 10$.

Hence, the total time of $12x = 120$ minutes or 2 hours.

It can be seen from the above problem that the use of the ratios involved makes it unnecessary to convert from one unit of measurement to another in order to determine the answer. This is often a considerable saving.

It can also be clearly seen that any two speeds that result in the ratio of 7:5 or any ratio where the numbers involved are two apart will yield the same value for x as in the problem given. Students understanding this fact can make up any number of new problems which would yield the same result for x. Nor need they keep to the

same unit of feet per second for the rate. Yards per minute, miles per hour, or rods per month could have been used equally well as long as the ratio was 7:5, or 8:6, or $5\frac{1}{2}:3\frac{1}{2}$, or any ratio with numbers whose difference was two. In all cases, the denominate units used are eliminated in the process of taking the ratios, thus making it entirely unnecessary to convert the units being used for the different headings.

We are now ready to summarize the values to be gained in the application of variation to problem-solving:

1. The mathematical powers of a student are developed through the process of using one solution for many problems of a related type. Consider the possibilities that exist for magnifying the range and scope of a student's efforts. Through such a process, he gains an insight into the inter relationships of problems. He has as a result a powerful means of integrating the subject matter of mathematics as well as a means of correlating mathematics with science or any other field that utilizes mathematical procedures. Furthermore, having gained an understanding of the methods employed in the solution of one kind of problem, these interrelationships enable him to solve the related kinds of problems. The problem that concerns the average rate of return on more than one investment; the one that treats of the average rate of speed; the average price of a mixture; the balance point of a lever; and the percentage of a chemical in a solution; all these, because of their mutual interrelationship as members of the $xy=z$ family, have exactly the same mathematical procedure and can be transformed from one to the other.

2. The creative abilities of the student find ready expression in the treatment of interrelated problems. The data used in one can be utilized in the other. Students should be encouraged to design problems in allied fields where similar formulas are used. New problems can be created which differ materially from those currently offered in mathematical texts.

3. The use of ratios enables the student to suggest any number of illustrations of problems whose solution will be exactly the same as the problem being treated. The avoidance of unnecessary manipulation means more time for concepts, understanding and problem-formulation.

4. The obtaining of ratios eliminates decimals and units of measure. In addition, their use often results in the replacement of larger numbers by much smaller ones where the larger numbers have a common factor.

5. In the resulting equation, the use of ratios usually yields a proportion affording easy solution. Quite often, the fractions are frequently circumvented as contrasted with traditional methods. Note the two solutions in the original "alcohol" problem.

6. In all cases of direct and inverse variation, the constant element need not be involved in the resulting equation if ratios are used. As a consequence, the equation can be stated in a simpler form.

Before proceeding to the second phase of my paper, I would like to clarify the procedure which I have used in the earlier part of my talk. It has been my conscious purpose to use few problems for illustration rather than many to bring out the conclusions I have stated. Although the examples provided are all of the inverse variation type, they could just as readily have been of the direct, or joint variation kind. This idea will be elaborated upon in the latter part of my talk when these three types of variation are shown to be different aspects of every "factor times factor equals product" formula or problem, or any formula which, because of additional factors, is an extension of the $xy=z$ type.

The second phase of this discussion concerns the problem of how we can teach variation so that the values of the subject may be realized in the classroom.

What is the correct approach to the subject? The following method is the one that I have found most satisfactory in my own teaching of the subject. You may find it suitable for your needs.

After the students have become familiar with several related types of problems, the formulas involved should be listed in a column:

$$rt = D$$
$$rp = I$$
$$fd = n$$
$$lw = A$$
$$np = C$$
$$rs = A$$

These may then be symbolized in the general equation $xy = z$, where z is the product, and x and y are the factors. At the very outset, it must be made clear to all that *any property discovered for this general equation must be equally true for each and every formula of the $xy = z$ type.*

Sets of values are then obtained and paired as follows:

x	y	z
3	8	24
$\frac{1}{3}$	$\frac{4}{1}$	$\frac{4}{3}$
9	2	18

becomes

Rate	Time	Distance
3	8	24
$\frac{1}{3}$	$\frac{4}{1}$	$\frac{4}{3}$
9	2	18

Hence, a person who travels $\frac{1}{3}$ as fast as another and takes four times as long will go 4/3 as far.

x	y	z
$\frac{4}{3}$	$\frac{3}{4}$	$\frac{1}{1}$

becomes

Length	Width	Area
$\frac{4}{3}$	$\frac{3}{4}$	$\frac{1}{1}$

Included in each table are the ratios of the corresponding elements in any column. Note that no distinction is made in the beginning between cases where all the Direct or Inverse Variation. Nor should there be any impatience on the part og the quantities vary and those where one of the elements remain constant, such as in

Or the ratios alone might be used:

x	y	z
3	8	24
$\frac{1}{3}$	$\frac{4}{1}$	$\frac{4}{3}$
9	2	18

x	y	z
6	14	84
$\frac{1}{1}$	$\frac{7}{1}$	$\frac{7}{1}$
6	2	12

x	y	z
12	15	180
$\frac{4}{3}$	$\frac{3}{4}$	$\frac{1}{1}$
9	20	180

Thus, in the case of two rectangles whose area is the same, if the length of the first is 4/3 as much as the second then its width is only 3/4 as much.

The student should now notice that whenever a quantity remains constant, the ratio equals 1; or conversely when the

ratio is 1, the values of the quantity in question are constant or remain the same.

We are now ready to develop the relations between the three types of variation in connection with $xy=z$.

Where all three quantities vary in $xy=z$, the resulting relation between the ratios of the three quantities is expressed in the equation, (x-ratio)(y-ratio) equals z-ratio. This result would be the same if there were additional factor-constants on either side of the $xy=z$ formula. For example, if $xy=kz$, or $abcxy=kz$, assuming that a, b, c, and k are constants, the ratio relation would still be the same, inasmuch as the ratio for values of a constant is always 1.

Whenever the (x-ratio)(y-ratio) = z-ratio z is said to vary jointly as x and y. Hence, since distance equals rate times time, if no one of the elements is constant, distance varies jointly as the other two. Or, since the volume of a rectangular box equals the product of the length, width, and height of the box, if any one of the three dimensions remains constant, the volume is said to vary jointly as the other two, and the ratios for any two sets of values would be in the (x-ratio)(y-ratio) = z-ratio form.

Direct and inverse variation should now be considered as special instances of joint variation. To obtain direct variation or inverse variation, it is merely necessary that one of the three elements remain constant. If the product z is constant, we will have inverse variation. If one of the two factors x or y remains constant, direct variation will result. Let us consider these two special forms of variation in detail.

Inverse Variation

If z remains constant, then

$$(x\text{-ratio})(y\text{-ratio})=1$$

or

$$x\text{-ratio}=\frac{1}{y\text{-ratio}}.$$

Since the x-ratio is the inverse of the y-ratio, we say that "x varies inversely as y." Clearly, in this case, "y also varies inversely as x."

Direct Variation

If x remains constant, then

$$y\text{-ratio}=z\text{-ratio}.$$

Since the two ratios are directly equal, we say that "y varies directly as z." It is evident that if y remained constant, "x would vary directly as z."

The three types of variation should then be applied to the many types of problems. For each formula, the student should show when any one of the three types of variation would occur and the resulting ratio relationship.

For example, where $A=lw$, students should be able to state not only when any one of the three types of variation would result but also what the relation would be between the ratios of the variable elements.

Extension to formulas with more than two factors is easily made if the same treatment is carried out with the abstract equations $wxy=z$ or, if necessary, $vwxy=z$. The ratio relationship evidentally is of the same form as the value relationship except for the fact that the constancy of any of the elements results in the replacing of its ratio by 1, thus reducing the number of ratios to be considered. Hence, if $vwxy=z$ and if v and z remained constant, then (w-ratio)(x-ratio)(y-ratio) = 1. Students should state the conditions needed to secure direct or inverse variation.

Applications may be made of formulas such as $V=lwh$, or $V=s^3$, $A=6s^2$, or $I=prt$.

1. For $V=lwh$:

If V remains constant, and the length and width are both doubled, then the new height will be one-fourth of the old height.

2. For $I=prt$:

If p remains constant and r is doubled and t quadrupled, then I will be eight times as large.

3. For $A = 6s^2$:

If s is tripled, then A will become 9 times as large. This follows from the fact that the A-ratio equals the square of the s-ratio.

4. For $V = s^3$:

If V becomes 1000 times as large, then s will only be ten times as great. In this case, the V-ratio equals the cube of the s-ratio.

5. For $F = kMm/d^2$:

If k, M, and m are constant, then if d is tripled, the force (F) will be only one-ninth as large, inasmuch as the F-ratio $= 1$ divided by the square of the d-ratio in this instance.

The proof of the ratio relationship is a simple one and will provide added understanding of the principle involved.

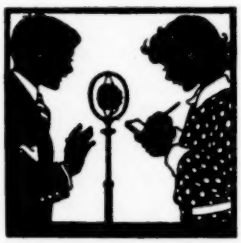
If $xy = z$, then $x_1y_1 = z_1$ and $x_2y_2 = z_2$ where the letters with subscripts are special values of the quantities x, y and z.

By division,

$$\frac{x_1y_1}{x_2y_2} = \frac{z_1}{z_2} \quad \text{or} \quad \frac{x_1}{x_2} \cdot \frac{y_1}{y_2} = \frac{z_1}{z_2}.$$

That is, the (x-ratio)(y-ratio) $=$ z-ratio.

The extension of the proof to cases with more than two factors is, of course, obvious.



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Direct, Practical Experiences in Mathematics and Success in Solving Realistic Verbal "Reasoning" Problems in Arithmetic

By W. J. Lyda

Morgan State College, Baltimore, Maryland

In this article, the writer will describe an investigation in which he attempted to determine the potency of direct, practical experiences in mathematics as distinguished from classroom activities as a conditioning factor in solving realistic verbal "reasoning" problems in arithmetic. This investigation is based upon three fundamental assumptions, namely:

1. Intelligence and mathematical experience are presumably conditioning factors in success in solving realistic verbal "reasoning" problems.
2. Most of our students in our classes in junior high school are average in intelligence.
3. Our text books are supposed to be written for the average student.

In order to select a group of thirty realistic verbal "reasoning" problems in arithmetic, the writer made an analysis of 1534 such problems as found in the arithmetics used in the fifth, sixth, and seventh grades of the colored schools of Augusta, Georgia, in terms of the relationship of the situation of each problem to one of the following objectives of education: health, leisure, social living, and consumer-vocational education. Next, a representative sampling of 150 problems was submitted to ten junior-senior high school teachers who rated each problem on the basis of how realistic she considered it. Third, average ratings were computed for each problem. Finally, upon the basis of these average ratings the writer selected the thirty most realistic problems.

Following the selection of the group of thirty realistic verbal "reasoning" problems in arithmetic, the investigator constructed a mathematical experience check list which was based upon and paralleled the situation of each problem. Some examples of the mathematical experiences of the check list are as follows:

1. Comparing the number of pages of library reading you do with that of a friend.
2. Depositing money in a bank.
3. Making a recipe smaller.
4. Reading the speedometer on an automobile on beginning a trip and at the end of the same trip.

After locating the students at the seventh grade level who were below average, average, and above average in psychometric intelligence, the writer administered his mathematical experience check list by the use of the group-interview technique to these students and they, in turn, indicated whether or not they had had a given mathematical experience often (more than twice), seldom (once or twice), and never by putting a check mark in the appropriate space on the check list. Next, he computed the percentages of students in each intelligence category who stated that they had had a given experience often, seldom, and never.

The group of thirty realistic verbal "reasoning" problems was then administered as a written test to these same pupils. The percentage of seventh grade students average, below average, and above average in intelligence who worked each of these problems correctly, was computed.

In analyzing the percentages of pupils—below average, average and above average—in psychometric intelligence who indicated they had had certain mathematical experiences often, seldom, and never and the corresponding percentages of pupils in each of those groups who worked each of those problems upon which such experiences were based, the writer confined himself to those problems which were worked correctly by at least 75% of the pupils of

each intelligence grouping. In this connection, it should be remembered that the situations of the check list were based upon and paralleled the situations of the problems.

Upon the basis of this scheme of analysis, the writer observed the following facts with respect to the pupils who were below average in intelligence:

1. On four of the thirty problems, 75% (or more) of the pupils worked the problem correctly.
2. On three of these four problems, a greater percentage of pupils who stated that they had had the mathematical experience based upon it often rather than seldom or never, worked the problem correctly.
3. On the other problem, a greater percentage of pupils who had the mathematical experience seldom rather than often or never, worked the problem correctly.
4. In all cases, the students had had the mathematical experience which was based upon and paralleled the situations of these problems, either often or seldom.

With respect to the pupils who were average in intelligence, the writer noted that:

1. On eleven of the thirty problems, 75% (or more) of the pupils worked the problem correctly.
2. On seven of the eleven problems, a greater percentage of students who stated that they had had the mathematical experience often rather than seldom or never, worked the problem correctly.
3. On two of the problems, a greater percentage of pupils who had had the experience seldom rather than often or never, worked the problem correctly.
4. On the other two problems, a greater percentage of students who said they had never had the mathematical experiences worked the problem correctly than those who had the experience seldom or often.
5. Therefore, on nine of these eleven problems, the greater percentage of students who had had the mathematical experiences which were based upon and paralleled the situations of these problems, respectively, than those who had not, worked these problems correctly.

Concerning the students who were above average in intelligence, the writer observed that:

1. On fourteen of the thirty problems, 75% (or more) of these students worked the problems correctly.
2. On six of these fourteen problems, a greater percentage of the pupils who stated that they had had the mathematical experience often rather than seldom or never, worked the problem correctly.
3. On five of the fourteen problems, a greater percentage of the students who indicated that they had had the experience seldom rather than often or never, worked the problem correctly.
4. On three of the fourteen problems, a greater percentage of pupils who indicated that they had had the experience never rather than seldom or often, worked the problem correctly.
5. On eleven of the fourteen problems, a greater percentage of pupils who had had the mathematical experience based upon and paralleling the situations of these problems, respectively, than those who had never had such experiences, worked these problems correctly.

Upon the basis of the facts presented in the preceding paragraphs, the writer arrives at the following conclusions:

1. Direct, practical mathematical experiences, as distinguished from classroom activities, which are related to the situations of realistic verbal "reasoning" problems in arithmetic is a potent factor in success in solving such problems.
2. Direct, practical mathematical experience based upon the situation of a realistic verbal "reasoning" problem in arithmetic is a more potent conditioning factor in success in solving such a problem for students below average in intelligence than for students average or above average in intelligence. Similarly, it is a more potent conditioning factor for students average in intelligence than for those who are above average in intelligence.

National Council Yearbooks!

The National Council Yearbooks are rapidly going out of print. This issue (last page) contains a list of those still available and the prices for which they may be obtained. Teachers who wish complete files and, particularly, school libraries who lack certain books should order now.—EDITOR.

School or Packing Mill

By Commander A. G. Norwood
Brooklyn, N. Y.

Early in 1939 a conference was held in Washington to plan the naval air expansion program. When the picture was painted of air bases to be scattered over the face of the globe in undreamed-of numbers, carriers at sea multiplied by ten —no, by 100!!—the problem was clearly that of manning this air arm.

The Civilian Pilot Training Program had prepared a reservoir of thousands of pilots, and it was felt that the colleges could be counted upon to furnish all the needed pilot material. However, for every man at the controls of an airplane there would be need for at least *ten* highly trained men on the ground: Machinist's Mates, Metalsmiths, Ordnancemen, Radiomen, Radar Operators, Aerial Photographers, Electrians, Instrument Technicians, and so on through countless trades and specialities. *Where were these men to come from?*

The census indicated that there were millions of men between 18 and 25 years of age who had graduated from high schools, and other millions who had had two years or more of secondary education. Educators assured the world that these men could be trained quickly in "speciality schools," but it was admitted that the Navy should set up its *own* schools to teach the specialized combat skills and trades.

Trade schools, or taining centers, were established at a number of naval air stations and production schedules were worked out in order to man the squadrons as they were formed. Barracks, classrooms and shops were built to handle the required complement of trainees while trained men from the fleet were transferred to duty as instructors.

A battery of "I.Q." and "Mechanical Aptitude" tests was given to the recruit training stations for use in screening the men to enter these schools, and a curriculum was developed for each school, predicated upon the assurance given by the "Educators" that high school students had been trained in following written orders and in working out the simple problems to be encountered in specialized training.

Instructors were assigned and lesson plans were developed. Equipment was mounted, "Ship's company" was rehearsed, and the training program was "established."

Within the first few weeks of training, it became painfully apparent that the high school boy could *not* work out the problems in simple shop mathematics needed in his war-time duties, and the twelve week course had to be increased to one of *sixteen* weeks. The schedule of graduation had to be maintained inasmuch as the formation of the new squadrons depended on the *entire* crew being on the line *on time*. The change from twelve to sixteen weeks in the course increased the trainee complement of each school by 33%, thus necessitating new barracks, mess halls, and gallies as well as additional instructors and Ship's company men.

As the first few classes progressed, it was found that the average trainee was not only totally incapable of interpreting simple working drawings and data sheets, but that he seemed unable to read and *understand* any written order (other than one pertaining to "liberty") and that his basic knowledge of simple science as related to mechanics (levers, inertia, momentum, etc.) was conspicuous by its absence.

This forced the sixteen weeks course up to twenty-one weeks, with all the additional problems of greater enrollment. For instance, one school had been built to meet a schedule of 800 graduates per week on

the twelve-week schedule and was designed, therefore, to handle 9600 trainees, but under the new term of twenty-one weeks, the school was built up to *16,800* trainees. With the consequent increase in instructors, masters-at-arms, commissary and supply forces, the total of naval personnel very nearly approached 20,000.

The first *big* question that stood out in the *glaring* light of our efforts was: "*Does the public school provide its students with the basic training that will enable them to think, or does it teach them to memorize a mass of more or less related facts?*"

In an effort to find the answer to the above question, we developed two *one-question tests* to be given to 12,000 trainees selected at random, the first test given on entrance and the second after sufficient time for the actual *figures* in the first to be forgotten.

In Test 1, the student was given a mimeographed sheet with the following question $5/8+1/64-1/32=?$ (Note that this is stated in exactly the same manner as a classroom blackboard problem.) Of the 12,000 men taking the test, 88% solved the problem correctly. Test 2 read as follows: "You are required to drill a hole for a 5/8" clevis pin; as this is to be a rotating, or "bearing," fit, it is necessary to ream out 1/32" of the metal for smoothness. The engineering manual requires this clevis pin to have 1/64" clearance when installed. What is the size of the original hole to be drilled?" To the eternal shame of the public schools, only 10% of the 12,000 students could solve it.

On going further into this phase of "public education," it was found that the phenomenon of students reaching high school, and *going through* high school without the ability to read, and *understand* the English language beyond sixth grade levels is so frequent that it excites little or no comment in educational circles.

This is not the fault of the student, for he wants to know what is going on, as evidenced by the preponderance of people reading the tabloids, for only in the tabloid press can they escape the frustration that comes with lack of comprehension of the printed word, and yet through the medium of pictures with headlines, become aware of the state of the world.

During the five years of war training (when my school alone graduated over 100,000 men for the fleet), I visited public high schools in all sections of the country, from the small district cooperative to the giant marble maze of the city school, and I became convinced that "mass education is *not* education of the masses."

It is a common thing to find a beautiful stone high school, set high on a hill for all the residents to point out with pride, with bright sunny classrooms, laboratories and shops shining with steel and glass. *However*, the beautiful classrooms for *twenty* students are crowded with thirty-five to forty, the cafeteria is used as a study hall, and the harassed, generally underpaid teachers are so bound by the order to "let the student express his individuality" and so awed by the system of "grade by age" placement that they merely "get rid" of each class by passing them gracefully upward until they become "high school graduates."

A recent report of the New York State Board of Regents states that "large numbers of high school graduates are seriously *deficient in the basic tools of learning*," and the "New York Times" in a test given in 1943 found specific evidence to support this statement.

Following the lead of the "New York Times," we in the navy school found that of 16,000 students tested *62%* could not name two functions of Congress; *58%* could not name two of the freedoms guaranteed by the Bill of Rights; *80%* could give no clear indication of the contributions to our history of Thomas Jefferson and, believe it or not, *8%* had no idea who he was; *36%* could not outline the duties of the Mayor and Board of Aldermen in their own city, and less than *10%* could determine the income from $1250 for 18 months at 4% per annum.

The trainees all agreed that the "interest motivation" school of thought, so much in vogue in the high schools, gives them a perfect "out" for required reading inasmuch as they had merely to state "I did not find it interesting" to silence the teacher, and that the "grade by age" system of promotion coupled with the teachers' panicky fear of "excess failure" will insure them a diploma if they refrain from murder and arson for four years of attendance.

How *can* an "educator" keep a straight face while saying that the student is an immature youth and must be guided through the maze of learning, and in the same breath, by adherence to the "interest motivation" theory, make him the supreme arbiter of his mental destiny and by so doing assist in the job done in the home, of *completely* destroying mental discipline?

Teachers in thousands of schools have been surprised when their ne'er-do-well students have visited the school on furlough from the service, and it is discovered that they have mastered various skills (as evidenced by the stripes on their sleeves) more efficiently in a few months of *training* than in several terms of *"education."*

The public school is vocal in its silence as well as in its pronouncements. It *cannot* continue to evade the issue by pretending that it is not within its province. The false god of "classroom routine" must give way before the fundamental concept of *teaching*, and only then can we save our youth from the shock of incomprehension, of constant bewilderment in the face of inescapable mental problems that leave wounds that can be healed only by the ugly scar tissue of indifference.

One cannot come into administrative contact with hundreds of thousands of "Run of the mill" high school students without being painfully conscious that the schools are withholding some vital elements from the mind, with the result that the youth is developing mental scurvy.

When are we going to dismantle the sausage factory where the young mind is stuffed with a heterogeneous mass of facts and substitute a school where he is *taught to think?*

Commander A. G. Norwood was an Educational Consultant with the Civil Aeronautics Authority at the start of the recent National Emergency. In this capacity he had no small part in the concept and development of the College Pilots Training Program of 1937–1939.

The early spring of 1941 found him as Officer-in-Charge of the Aviation Machinist's Mate School in the Trade School Section of the Naval Air Station, Jacksonville, Fla. During his tour of duty on this assignment the school grew from 500 to 5000 trainess, the course of study was increased from 12 to 21 weeks, and a teacher training course was established. This teacher training course made it possible for the original staff of civilian instructors borrowed from public schools and industry to be replaced by a trained enlisted faculty, wise in the ways of the Navy as well as in *teaching*. Best of all, over 10,000 trained "Mechs" were sent to the far flung fleets.

In 1943 Commander Norwood was sent as Executive Officer to a new aviation training center that was destined to become one of the largest in the Navy, a varietal Naval University having within its boundaries a Machinist's Mate School, a Metalsmith School, an Ordnanceman School, a Radar School, a Rubber Accessory School, and a preliminary, or elimination, school for Aerial Gunners.

This training school grew from a trainee enrollment of a few thousands to one of 20,000 men and women, and included the first course in which Waves and Marine women were trained as Aviation Mechanics and Metalsmiths. At the peak of the program there were over 1000 women in training.

Inasmuch as the Executive Officer of a Naval Station has under his direct super-

vision *all* the activities of the station (from the price of cokes in the canteen to the curriculum in the schools, from the supervision of the Wednesday night dance to the promotion of instructors and officers, and from the approval of a "watch bill" for the fence guards' *horses* to the language used by Instructor Zilch in his class in flame propagation), it can be seen that he *should* know whereof he speaks.—EDITOR.

Geometry Jingles

By IVA D. OMAN, *Teacher of Mathematics, Topeka High School, Topeka, Kansas*

A crippled Scalene Triangle went limping with his cane
And met a jolly Circle upon a level plane.

"How do you do?" "And how are you?" they said like you and me
And then sat down together 'neath an old geome-*tree*.

And while they loitered in the shade they told the village news:
Right Triangle's baby crawls on its hypotenuse.

The Perpendiculars, they say, had quadruplets last night
And though it's 'most too soon to tell, they ought to be all right.

Obtuse Angle's home from school and with his parents stern.
He's not acute, he isn't right. He's just too dull to learn.

The Parallels are pleasant twins who never fuss or fret
Though they must act like strangers because they've never met.

Someone said that Poly Gon had eaten too much cake
And her indigestion's awful with so many sides to ache.

One Quadrilateral, they say, will always treat you fair.
He's quite a regular fellow and he signs his name, "B. Square."

His cousin Rhombus won't sit tall but slumps down in his chair
And all the teachers frown and say, "I wish you'd act like Square!"

Scalene's cousin never limps but walks with greatest ease
Because his legs have equal length. His name's Isosceles.

A chap whose name is Trapezoid is shaped quite like a bin
Two sides are level parallels and two sides just slope in.

The Point thinks he's important though he hasn't any size.
He knows he has position and position satisfies.

But should he move, he'll make a streak, a streak so very fine
It hasn't either depth or breadth. They call the streak a Line.

And when a Line gets restless and moves from there to here
It generates a Surface like on a cube or sphere.

And if a Surface moves—Well! Well! Hang on and keep it steady
For the subject's getting solid and you're really not quite ready!

Soon Circle yawned and Scalene stretched and both sighed wearily
And in geome-tree's deep shade they slept like you and me.

FOR SALE

THE MATHEMATICS TEACHER still has a few unbound complete volumes of the magazine, beginning with 1925, for sale at $2 each postpaid. Write at once if your are interested.—EDITOR

More Thoughts on Placing the Decimal Point

By W. E. KING

Director of Mathematics, Fort Worth Public Schools, Fort Worth, Texas

MANY teachers of arithmetic probably enjoyed reading the article, "Some Thoughts on Placing the Decimal Point in Quotient," by Claude H. Brown in the February (1945) issue of THE MATHEMATICS TEACHER. Since a majority of the textbooks about arithmetic use the caret method in division of decimals, some teachers are of the opinion they should teach that method even though they prefer to teach the subtraction method. Many of them believe that the subtraction method has more meaning for their pupils. For these reasons they are glad that someone champions the subtraction method.

Most of the boys and girls who are graduated from the public schools have studied arithmetic for at least eight years. Yet, with all the time that was devoted to the study and use of numbers, some of them can not solve the easy problems encountered in ordinary business transactions. The public schools are sometimes criticized by business and professional men and by officials of the armed forces, and there may be some justifications for criticism. If public school graduates can not do effectively the fundamental operations in arithmetic and the problems involving these operations, something is wrong.

It seems that the major weaknesses in the arithmetic program are that pupils do not gain an adequate knowledge of the meanings inherent in the number system and that they do not understand the reasons and the purposes for the various operations with numbers. It is the writer's opinion that every operation used in working examples and in solving problems with numbers has a meaning, a reason, and a purpose and that normal pupils who have been taught well can understand and use these effectively.

Believing that the best method or device for performing any operation with numbers is the one that imparts the most real understanding of the functions of numbers, the writer prefers the subtraction method of dividing decimals to the caret method. For the same reason it appears that there is even a better method than the subtraction method and also a better method in multiplication of decimals than the addition method. The purpose of this paper is to present these methods. The presentations will, of course, be made with less detail than is ordinarily used with pupils.

These methods may or may not be new. The research that the writer has made fails to reveal their use in any textbooks or in any books on the teaching of arithmetic. Like the other methods they may have limitations. These methods are based upon place values in the decimal system of numbers. As is well known, numerals used in numbers have both inherent and place values. Ray[1] says, "The value of a figure in a number is local." If pupils understand the local values of numerals in numbers and the meanings and uses involved in the fundamental operations up to multiplication and division of decimals, they are ready, and not before, to learn and to use these methods. There are two steps in the presentation of each method and the first step in each should be a review for the pupils.

MULTIPLICATION OF DECIMALS

Step 1. Easy examples in multiplication of whole numbers are used to recall some basic facts. Each step should be analyzed for its meaning.

Illustrative Examples

1.

$$\begin{array}{r} 32 \\ \times\ 3 \\ \hline 96 \end{array}$$

[1] Ray's New Practical Arithmetic, 1877.

2.

$$\begin{array}{r} 213 \\ \times\ 32 \\ \hline 426 \\ 639 \\ \hline 6816 \end{array}$$

In these two examples the facts encountered are that the product of:

1. units times units is units
2. units times tens is tens
3. units times hundreds is hundreds
4. tens times units is tens
5. tens times tens is hundreds
6. tens times hundreds is thousands

Examples with larger numbers may be used to review the larger facts.

The place value of the product of any two numerals depends upon the locations of the two numerals in their respective numbers. For instance, the product of 2 units times 3 units is 6 units, the product of 2 tens times 3 tens is 6 hundreds, the product of 2 tens times 3 hundreds is 6 thousands, etc. When larger numerals are multiplied, the products may have two places as follows: the product of 4 units times 6 units is 24 units or 2 tens and 4 units, the product of 5 tens times 6 tens is 30 hundreds or 3 thousands and 0 hundreds. Pupils who are familiar with number placement should have no difficulties with such products.

Step 2. In this step four examples of increasing difficulty are used to present the multiplication of decimals:

1.

$$\begin{array}{r} .4 \\ \times\ 2 \\ \hline .8 \end{array}$$

2.

$$\begin{array}{r} .3 \\ \times\ .2 \\ \hline .06 \end{array}$$

3.

$$\begin{array}{r} 13.2 \\ \times\ .3 \\ \hline 3.96 \end{array}$$

4.

$$\begin{array}{r} 21.32 \\ \times\ 1.3 \\ \hline 6.396 \\ 21.32 \\ \hline 27.716 \end{array}$$

In the first two examples the pupils should use common fractions to make the place values of the numerals 8 and 6 meaningful, as 2 times $\frac{4}{10}$ is $\frac{8}{10}$, and $\frac{2}{10}$ times $\frac{3}{10}$ is $\frac{6}{100}$. The use of common fractions is advisable until the pupils gain complete understanding of the simpler decimal facts. In these four examples the decimal facts are that the product of:

1. units times tenths is tenths
2. tenths times tenths is hundredths
3. tenths times units is tenths
4. tenths times tens is units
5. tenths times hundredths is thousandths
6. units times hundredths is hundredths

In example (1) the decimal fact is tenths, and one decimal place is required in the product; in example (2) the decimal fact is hundredths, and two decimal places are required in the product; the same is true in example (3) while in example (4) the smallest decimal fact is thousandths, and three decimal places are required in the product. Therefore, it appears obvious as well as meaningful that the smallest decimal fact or the place value of the first numeral that is obtained in the product determines the location of the decimal point in the product. In using this method the rules for placing the decimal point in the product may be stated as follows:

(1) Determine the place value of the first numeral obtained in the product.
(2) Place the decimal point in the product so that this numeral has its determined place value.

Division of Decimals

Step 1. Easy examples in division of whole numbers are used to review some basic facts in division. Three examples are used:

1.

$$2\overline{)86}\quad 43$$

2.

$$\begin{array}{r} 12\overline{)4956} \quad 413 \\ 48 \\ \hline 15 \\ 12 \\ \hline 36 \\ 36 \\ \hline \end{array}$$

$$
\begin{array}{rr}
 & 5\,2 \\
2\,3\overline{)} & 1\,1\,9\,6 \\
 & 1\,1\,5 \\
\hline
 & 4\,6 \\
 & 4\,6 \\
\hline
\end{array}
$$

In example 1 the first quotient numeral is obtained by dividing 8 tens by 2 units. It is 4 tens because the quotient of tens divided by units is tens. The second quotient figure is 3 units because the quotient of units divided by units is units. In example 2 the first quotient numeral is obtained by dividing 4 thousands by 1 ten. It is 4 hundreds because the quotient of thousands divided by ten is hundreds. The remaining quotient numerals are obtained by dividing hundreds by tens and tens by tens, giving 1 ten and 3 units. In example 3 the first quotient numeral is obtained by dividing 11 hundreds by 2 tens and not by dividing thousands by tens, since the first two places in the dividend must be considered as divisible by 2. It is evident that the first numeral in a divisor gives it a place name, but the group of numerals considered at one time in the dividend may get a place name from the first numeral or from the second numeral as illustrated in these three examples. Additional examples may be used to review other division facts when needed.

Step 2. In this step new facts are learned, and from these facts a different method for placing the decimal point in the quotient is explained and made meaningful to the pupils. Four examples will be used:

$$
\begin{array}{rr}
 & 2.3 \\
1.\ \ 2\overline{)} & 4.6
\end{array}
\qquad\qquad
\begin{array}{rr}
 & .0\,2 \\
2.\ \ 2\,3\overline{)} & .4\,6
\end{array}
$$

$$
\begin{array}{rr}
 & 2\,3.1 \\
3.\ \ .3\overline{)} & 6.9\,6
\end{array}
\qquad\qquad
\begin{array}{rr}
 & 3\,4\,2\,0 \\
4.\ \ .2\,1\overline{)} & 7\,1\,8.2 \\
 & 6\,3 \\
\hline
 & 8\,8 \\
 & 8\,4 \\
\hline
 & 4\,2 \\
 & 4\,2 \\
\hline
\end{array}
$$

Before the pupils begin to work these examples their attention is again directed to the facts that in mixed decimal numbers the decimal point is always located between the units and the tenths and that places to the left and to the right of the decimal point derive their place names from their respective number of places from the decimal point. In example 1 the first quotient numeral is obtained by dividing 4 units by 2 units. It is 2 units because the quotient of units divided by units is units. Therefore, the decimal point must be placed after this first quotient numeral, and the next quotient numeral is tenths because it is the first place to the right of the decimal point. In example 2 the quotient numeral is 2, and it is in the hundredths' place because the quotient of tenths divided by tens is hundredths. This fact should be illustrated with common fractions, as $\frac{4}{10}$ divided by 2 tens or 20 is $\frac{2}{100}$. Therefore, a zero must be prefixed, preceded by the decimal point. In example 3 the first quotient numeral is 2, and it holds the ten's place because the quotient of 6 divided by 3 tenths is 2 tens or 20. The following places are units and tenths respectively. In example 4 the first quotient numeral is 3, and it holds the thousand's place because the quotient of 7 hundreds divided by 2 tenths is 3 thousands. When the division is completed, a zero is annexed to give the first numeral in the quotient its place value. The decimal facts learned in these four examples are that the quotient of:

1. tenths divided by units is tenths
2. tenths divided by tens is hundredths
3. units divided by tenths is tens
4. tenths divided by tenths is units
5. hundredths divided by tenths is tenths
6. hundreds divided by tenths is thousands
7. tens divided by tenths is hundreds

It was pointed out in the division of whole numbers that the first numeral in the divisor gives the divisor a name. Like-

wise, in the division of decimals the first numeral in the divisor gives it a name, as in example 3 the divisor is considered as tenths and not hundredths.

In presenting this method to pupils as many simple examples as are necessary should be used to give them an understanding that the place value of the first numeral that is obtained in a quotient must be determined and then the decimal point is placed in the quotient so that the first numeral holds its determined place value.

After this understanding is gained, the pupils should be able to state the rules in their own words. The rules may be stated in several ways. These seem to be as simple and meaningful as any other directions for placing the decimal point in the quotient.

1. Find the place value of the first numeral in the quotient.
2. Place the decimal point in the quotient so that the first numeral in the quotient has its determined place value.

Do these methods of placing the decimal point in the product and in the quotient have more merits than the other methods? They appear to have these points in their favor: (1) the pupils must have a thorough understanding of number placement; (2) each operation has a meaning from the place value viewpoint; (3) they can be used regardless of the relative positions of the numbers that are used in multiplication and division; (4) they can be used with computing devices; and (5) the rules are short and easy to comprehend.

At first it may appear that these methods require the learning of too many facts for the average pupil who is beginning to learn these operations. There are, however, many educators who believe that the arithmetic program gives too much attention and emphasis to multiplication and division of decimals and that the examples and problems are often difficult and impractical. The average person seldom uses decimals except in the solution of problems with United States money. It appears, therefore, that pupils who are beginning the study of decimals should not be required to multiply and divide with numbers having more than two decimal places. If these operations are presented in the most meaningful ways, and with *simple* examples and problems, the pupils should be able to apply the meanings to more difficult problems when the need arises and, also, be able to use devices and short cuts, if desirable, in all practical situations.

Enrollments in various types of institutions as of October 15, 1946, were as follows:

	Number of Institutions	Enrollment Total	Enrollment Veterans
Universities and large institutions of complex organization	131	1,017,000	585,000
Colleges of Arts and Science	557	439,000	195,000
Independent technical and professional schools	287	210,000	129,000
Teachers colleges and normal schools	201	150,000	62,000
Junior Colleges	468	187,000	84,000
Negro institutions	105	59,000	18,000
TOTAL	1,749	2,062,000	1,073,000

Hypotheses as a Point of Departure in Geometry Teaching

By KENNETH B. HENDERSON

University of Pennsylvania, Philadelphia, Pa.

THE word "hypotheses" is used synonymously with "given conditions" in many high school geometry texts. This procedure gives the students an acquaintance with only one of the meanings of the word. There is another sense in which the word is used which is significant enough to warrant its consideration. This sense is the one used by the scientist when he speaks of a possible answer to a problem. As such, it is one of the steps in the thinking process.

The use of the word in this sense in the geometry class makes possible a different treatment of the usual subject matter. Suppose the teacher in introducing the study of parallelograms should draw a parallelogram on the board, and, without naming it, ask the students to state the theorems concerning the parallelogram which appear to be true. The pupils may be expected to suggest two or three of the theorems generally found in the textbook. The drawing of the diagonals and the extension of the sides will suggest a few more. Let these statements be called the "hypotheses" or tentative theorems.

It remains to put these to a test to see whether they really can be substantiated by proof. It then becomes evident that one relationship in the figure must be known before anything can be proved. If the teacher cares to cut loose from the textbook, the students may be allowed to develop their own definitions of a parallelogram. Some pupils may define "parallelogram" in the usual way. Their proofs of the hypotheses would be similar to those given in the usual textbook. Others may choose to define "parallelogram" as "a quadrilateral whose opposite sides are equal" in which case their proofs of the hypotheses would be different although logically sound. The postulate of one pupil would be an hypothesis of another and vice versa.

It may even be possible that some students will define parallelogram as a quadrilateral one pair of whose opposite sides are equal and parallel. In this case he would prove as a theorem. "The opposite sides of a parallelogram are parallel," which is the basis of the conventional definition of a parallelogram.

Class discussions of proofs could center upon consistency and the absence of unwarranted assumptions rather than conformity to the text's sample proof. It is easily possible that a student might disprove a theorem stated by a classmate. The ability to disprove a statement or theorem, if this is possible, affords just as much opportunity for good reasoning and an air-tight argument as proving a theorem. It may well be that forever concentrating on proving theorems in geometry may give students the narrow conception that the technique employed is unsuitable for disproving a statement or contention.

Careful direction of the discussion should lead the students to get a picture of postulational thinking—that we must start somewhere with some postulates and definitions. We deduce theorems from these, others from these, and so on. Each deduced theorem is a function of the postulates.

Criteria may be developed cooperatively by the class and teacher to judge the significance of theorems (hypotheses) that appear to be true. The theorem, "The sides of a parallelogram are not necessarily all equal, but they may be," affords opportunity for proof even though it is a poor statement. It can be reworded by

employing the criteria for a satisfactory theorem.

Subtle guidance should result in each pupil's covering the same body of theorems (if such is considered necessary) but in a much more meaningful manner. Let the students actually construct their own bodies of theorems. As each new theorem pops into mind, it is regarded as an hypothesis. If it can be proven, it then is accepted as an established theorem or proposition.

The bodies of theorems will differ among each other in sequence and in methods of proof. The textbook is largely unnecessary. Each student defends his own proofs and forces others to defend theirs. Geometry is learned as it is made rather than by imitation of the "canned" proofs of the textbook.

True, this method will not cover the ground as rapidly. It may be inadvisable to employ it if different values are cherished. What it does accomplish in addition to what has been mentioned, is to place theorems, before they have been proved, in the same category as the inferences or hypotheses the scientist makes. They are discovered by insight the same as the scientist discovers his. They become valid if they can be conclusively proved. The difference is that the scientist relies chiefly on experimental corroboration while the mathematician demonstrates the theorem as a necessary consequence of other theorems, postulates, or definitions. Should the scientist's hypothesis fail when it is tested, it is discarded. Should the mathematician fail to demonstrate his hypothesis (theorem) to be a logical result of previously known theorems or be able to prove that it is not a logical result of such, this theorem is discarded. The important point is that in each case a test is made.

Approaching geometry from this angle detracts not a whit from the essential nature of the subject. It helps to emphasize the steps in thinking, one of which is to discover and test possible courses of action. It makes learning discovery rather than memory and puts the responsibility for discovery and testing on the shoulders of the student which is exactly where it ought to rest.

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The Retention by Teachers of Computational Skills in Arithmetic

By Oscar P. Rausch

Indiana University, Bloomington, Ind.

A study by Challman (1) reported in the February, 1946, issue of The Mathematics Teacher gave evidence that many of the fundamental skills in arithmetic are forgotten by students after they are no longer exposed to the environment of the classroom. Whether or not similar results would be obtained if a study were made to determine the retention of computational skills in arithmetic by teachers was a purpose of the following investigation.

The subjects of this investigation were 169 public school teachers. Of these, 125 are teaching in the secondary school, and 44 are elementary-school teachers. They are located in several states and represent many different schools. Their teaching experience ranges from a minimum of one year to a maximum of 31 years. All of them hold the Bachelor's degree and at the time of the investigation were enrolled in graduate courses at Indiana University. They teach in various areas of the curriculum. The secondary-school teachers were classified into the following groups according to their major teaching-field: English, mathematics, physical education, science, social studies, and a miscellaneous group. The latter group was comprised of teachers in fine arts, music, industrial arts, and languages. The number teaching in these respective areas was too small to enable classification into separate groups. It is apparent that no attempt was made to obtain a selected group.

The computational section of the *Stanford Advanced Test in Arithmetic** was administered to the group. This test was designed by the authors for the seventh, eighth, and ninth grades. The administrator of the test adhered strictly to the time limits prescribed by the authors.

The results of the test were quite interesting. The highest possible number of correct solutions to this section of the test is 65. The scores of the teachers who took the test ranged from 12 to 65. Only two teachers solved every problem correctly, and these two are teachers of mathematics. The average number of problems solved correctly by the total group was 44. Some of the errors were the results of carelessness in computation, but by far the majority of incorrect solutions were due to computational errors. The results for the various groups are shown in the following Table.

Table I

Average Number of Correct Solutions by the Various Groups

Groups of Teachers	No. of Teachers	Group Average*
English	21	38
Mathematics	17	58
Physical Education	16	41
Science	16	48
Social Studies	29	37
Miscellaneous	26	46
Total High School	125	44
Elementary	44	46
Grand Total	169	44

* To nearest whole number.

The results of the various groups probably rank about in the order that one would expect. It is apparent that the retention of computational skills is related to the use made of them by the respective teachers in the classroom. It should be noted, however, that about $\frac{1}{3}$ of the elementary teachers are instructors in the primary grades who do not teach arithmetic of the degree of difficulty that is found in the problems of this test. The average of the other $\frac{2}{3}$ of the elementary

* Kelley, Truman L., Ruch, Giles M., and Terman Lewis M., *Stanford Achievement Test.* World Book Company, Yonkers-On-Hudson, New York, 1941.

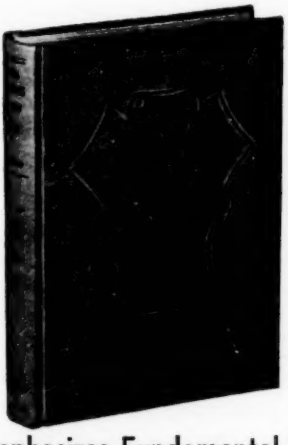
teachers would be considerably higher.

An analysis of the errors revealed that the difficulties encountered in the solving of the problems ranked in the following order of frequency: (1) Correct placement of the decimal point (2) Correct manipulation of the positive and negative signs in the fundamental processes (3) Ability to solve problems involving percentage (4) Ability to solve problems involving long division (5) Multiplication and division of fractions. Other errors occurred in varying degrees of frequency.

It is very apparent that many of the computational skills once learned are entirely forgotten, and that continued use is probably the major factor contributing to the retention of these skills. It seems very advisable to review arithmetic fundamentals in secondary schools and colleges before attempting to teach advanced courses requiring a thorough understanding and skill in the correct manipulation of numbers. To assume that students retain all the necessary skills from their elementary-school days, after a considerable lapse of time, is almost certain to invite disappointment and irritation for the teacher and increase the difficulty for the student.

REFERENCE

1. CHALLMAN, MILDRED, "The Retention of Arithmetic and Algebra in Relation to Achievement in Plane Geometry," THE MATHEMATICS TEACHER 39: 77–79, February, 1946.



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The Future of Visual Aids in Mathematics

By DONOVAN A. JOHNSON
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and
JAMES F. NICKERSON
University of Kansas, Lawrence, Kans.

A PERIOD of rapid expansion of teaching materials such as motion picture films, film strips, three dimensional pictures, mathematical instruments and models, both of an instructional and an enrichment nature, lies immediately before us. Are we going to give individual and collective aid in directing this expansion in the most desirable and efficient direction? The need for this direction is very evident to all who have seen some of the teaching aids that are now available in the field of mathematics education.

Most teachers of mathematics are agreed on the efficacy of visual aids in mathematics instruction and should now be ready to plan and execute a "Program of Action" to get these aids in use in every mathematics classroom. We tend to talk in terms of generalities and of the desirability of such materials, yet we fail to be specific and produce clearcut action, policy or plan. As individual teachers we want specific help as to sources, costs, specific methods and other guidance for the use of these aids. The eighteenth yearbook on "Multisensory Aids in the Teaching of Mathematics" is an excellent step in this direction. However, current developments need to be reported during this period of rapid expansion. In order to get good teaching aids into every mathematics classroom, there are two distinct needs to be met, namely:

1. To set up a long range outline of the present and future audio-visual needs of mathematics education and give every assistance to promote and direct the production of these teaching materials
2. Simultaneously to promote and develop the wide use of these teaching aids

One of the most immediate challenges before mathematics education lies in dedetermining this outline of present and future teaching needs in mathematics education. This could well be based on the report of the commission on Post War Mathematics. We must define the kind of film or model, equipment or picture we want for a specific teaching situation. Do we want motion pictures, films or film strips? Do we want informational materials or interest materials? Do we want traditional topics or the newest applications? Do we want pictures or diagrams? Perhaps it is all of these that we wish, but in what order should they be produced? What length? What type? For what purpose?

These and other problems need definition and outline. It is too large and important a task to leave to opportunistic venture. Mathematics teachers must be interested and organized in plotting the future of this important development in education.

There seem to be two alternatives before mathematics educators and commercial producers in this expansion.

1. Either set up a planning group to work out such a blue print cooperatively or
2. Let the development of future materials follow its expensive trial and error course. In this alternative commercial producers will hire the expert opinion of members of our profession and will attempt to estimate these needs as best they can and add the cost of experiments and failures to the cost of future teaching aids.

We prefer the first alternative, namely, a cooperative planning enterprise be started between educator and producer to determine these needs, the order of production, the prospective market, and the practicality of certain media for these needs.

The market for the use of visual aids in mathematics can be developed best by means of a systematic and reliable review and evaluation by a representative committee of mathematics educators and such specialists as are needed.

This review and evaluation should cover current educational films and teaching materials as released and be widely publicized. Information regarding the specific use of films, purchase and care of equipment, source of materials, suggestions for design of library facilities, operators clubs, purchase plans, methods of teaching with visual aids, and other necessary details of systematic use of visual aids must be made available to every mathematics teacher through many channels such as professional journals and special bulletins and pamphlets used by mathematical organizations and commercial producers.

We are at the threshold of the advancement of powerful teaching aids. We must guide its development as well as ready our own ranks to receive and use it to the fullest extent possible.

A Strong America Is a Peaceful America

FACT SHEET—ARMY DAY AND ARMY WEEK—APRIL 1947

DATES: Army Day will be observed on Monday, April 7. And this year, for the first time, the War Department will celebrate "Army Week,"—beginning April 6 and running through April 12.

PURPOSE: The purpose of Army Day and Army Week programs is to honor America's soldiers, living and dead, who did so much to make peace possible; to call attention to the new peacetime pattern of national defense which is being developed; to make the public aware of the Army's assignments both at home and abroad; to explain the need for a well-trained, efficient Army of volunteers adequate both in size and quality to discharge its designated duties as this country joins all United Nations in building permanent world peace; and to bring the people and their Army closer together in our national community.

THEME: "A STRONG America Is a PEACEFUL America." Every citizen must know of the relationship between an adequate Army at this time and the national security. Every citizen must be made aware of his personal responsibilities. Young men, in particular, must know of the opportunities for a career offered by the service in the continuing campaign to enlist thousands of volunteers a month. Army Week will stress not only the Regular Army but also the importance of new plans for the whole defense establishment—including the National Guard of the United States, the Organized Reserve Corps and the Reserve Officers' Training Corps. Plans now are under way to build these to strength, for training as manpower reserves for emergency.

This job really is a continuation of the tremendous conversion task done in the brief eighteen months since V-J Day. Eight million wartime veterans have been discharged and a real start has been made on the job of building a stable, efficient new Regular Army of more than a million men. At the same time, the Army has resumed its peacetime assignments at home, and it has been making scientific advances at a rate unprecedented in military history.

PROGRAM: The War Department will conduct many activities nationally to support Army Week, including national advertising and radio programs, public statements and the dissemination of public information. But most of the activity will be conducted locally, in the cities and towns of the country. During the week, Ground Forces and Air Forces will put on demonstrations and exhibits. Army posts and installations and National Guard armories will have "open house" on certain days (to be announced locally). And many patriotic, civic and fraternal organizations will join with the Army to celebrate the occasion.

As has been the case since 1928, when it was started on April 6, the anniversary of this country's entry into World War I, Army Day this year is sponsored by the Military Order of The World Wars. Because the date falls on Easter Sunday, the celebration will be observed on Monday, April 7.

◆ THE ART OF TEACHING ◆

Logical Approaches to $(-a)(-b)=ab$ and $x^0=1$

By H. Van Engen

Iowa State Teachers College, Cedar Falls, Ia.

An examination of ninth grade algebras shows that authors introduce the multiplication of positive and negative numbers in a variety of ways. This is particularly true of the product $(-a)(-b)$. The methods used to introduce $x^0=1$ are not as diverse as those used to introduce $(-a)(-b)$. However most of the methods used by elementary texts for introducing these two topics can be characterized as being distinctly pseudo-mathematical. Many texts appeal to higher authority (the "mathematicians"). A very few texts approach logical rigor. The authors of the latter texts protect themselves against charges of being illogical by pointing out that such exercises as: if the thermometer fell 3° per hour for the past 2 hours, then 2 hours ago it read 6° higher than now; that is $(-3°)(-2)=+6°$, are given for purposes of illustration only.

Now there are two logical approaches that can be used in the ninth grade to introduce the law of signs for multiplying positive and negative numbers. The same is true for non-integral exponents. These are (1) the definitional approach and what will be called for the purposes of this paper (2) the postulational approach. A modified version of the former approach has been used in some texts.[1] The latter approach, it will be contended, is so simple and psychologically feasible that more use should be made of it in beginning high school algebra classes.

Mathematically there is nothing new or startling in the postulational approach. In fact it necessarily is very elementary. Its publication can only be justified on the grounds that it may encourage high school teachers to rethink instructional procedures and to call to their attention the feasibility of a mathematically sound approach to $(-a)(-b)=ab$ which has not been used in high school textbooks. Any substitute for the so-called pseudo-mathematical approaches to $(-a)(-b)$ should be of interest to teachers since pseudo-mathematical approaches are not satisfying to either the teacher or the pupil. As evidence to support such a statement consider how often teachers, in desperation say, "Just remember a minus times a minus is a plus."

The Definitional Approach to $(-a)(-b)=ab$

This type of approach may be briefly illustrated by the following: $(+2)(+3)=(+3)+(+3)=+6$. Therefore $(+2)(+3)=+6$, and $(+2)(-3)=(-3)+(-3)=-6$. Therefore $(+2)(-3)=-6$.

Furthermore in arithmetic it was noticed that $(3)(2)=(2)(3)$, so $(-2)(+3)=(+3)(-2)=(-2)+(-2)+(-2)=-6$. Therefore $(-2)(+3)=-6$.

The above set of equations are typical of the equations that can be found in almost all elementary algebra texts. The critical teacher will notice that there is a hidden assumption in a set of illustrations of this type. The authors assume that negative numbers obey the rules that have been laid down for unsigned numbers. The teacher who is interested in developing a taste for the logic of the subject will avoid hiding assumptions in proofs. Some teachers even resent such procedures.

[1] See for example Hart, W. L., "Progressive First Algebra," D. C. Heath & Co., 1943.

There now remains only the case in which both factors are negative. Mathematicians have agreed that $(-2)(-3)=6$. They (the mathematicians) accept this fact without proof.

The above is not as rigid a definitional approach to the product of two negative numbers as the one preferred by mathematicians. However it is probably as close to such an approach as is possible in a ninth grade class. Notice that rather than state that $(-2)(-3)=6$, by definition, the authors usually appeal to the authority of the mathematician. This is a very natural procedure since it furnishes the teacher with a better "psychological in" with the student than would be possible by simply stating that $(-2)(-3)=6$ by definition. However, many teachers do not like to tell the student, "Just remember that the mathematicians have agreed to make a minus times a minus a plus." Even though this response does enable the class to proceed with the fundamental operations involving negative numbers it often does not produce a desirable mental set in some pupils. This is attested to by such oft-heard questions as "How do they know it will be right?" and "Can they call it whatever they want to?" All such questions are evidence that the definitional approach may be a disturbing one for the pupil.

The Postulational Approach to $(-a)(-b)=ab$

In order to make this approach possible it will be necessary for the teacher, and text, to take some time during the first part of the course to teach the "rules of computation." The class should learn to recognize that $ab=ba$ and that $a+b=b+a$. In words these symbols mean that multiplication and addition are commutative operations. Furthermore, the pupils should work with the associative property of multiplication and addition long enough so that they know what associativeness means and how to use it. In symbols, the pupil should recognize that $(ab)c=a(bc)$ and that $(a+b)+c=a+(b+c)$. And lastly that the pupil should know that multiplication is distributive with respect to addition and subtraction. That is $a(b\pm c)=ab\pm ac$.

With a sure foundation in the "rules of computation" (a few books do an excellent piece of writing when developing the "rules of computation") it is possible to present the multiplication of two negative numbers to a ninth grade class somewhat as set forth in what follows.

We have studied negative numbers and we know what they mean. We have added and subtracted negative numbers and have been requiring these new numbers to obey the same "rules of computation" that positive numbers and the unsigned numbers of arithmetic obey. In learning to multiply numbers we are going to continue requiring the negative numbers to obey the same rules of computation. Let us see what will happen if we assume that they do obey these laws.

1. $2[(+4)+(-3)]=2$ because the quantity in the bracket namely $(+4)+(-3)$ equals 1, and $2\times 1=2$.

But multiplication is distributive with respect to addition. Therefore, $2[(+4)+(-3)]=2(+4)+2(-3)=8+2(-3)$.

This last sum must equal 2 by 1 above. Therefore $8+(2)(-3)=2$.

Hence $2(-3)=-6$.

Now $(-3)(+2)=(+2)(-3)$ because we are requiring negative numbers to obey the computative law of multiplication. But it has already been shown that $(+2)(-3)=-6$. Therefore $(-3)(+2)$ will also equal -6.

What does $(-2)(-3)$ equal?

2. $(-2)[(+4)+(-3)]=-2$ because the quantity in brackets again equals 1, and $(-2)(1)=-2$. However $(-2)[(+4)+(-3)=(-2)(+4)+(-2)(-3)$ because we are requiring that negative numbers obey the distributive law. Continuing we have $(-2)[(+4)+(-3)]=(-2)(+4)+(-2)(-3)=-8+(-2)(-3)=-2$ (See

2 above). In order that $-8+(-2)(-3)=-2$ the product $(-2)(-3)$ must equal $+6$.

The teacher should note that this is a proof that $(-2)(-3)=6$ which can easily be generalized, for adults, to $(-a)(-b)=ab$. Notice further that when it was assumed that negative numbers were required to obey the "rules of computation" it was possible to prove that a minus times a minus is a plus.

The assumption that the negative numbers obey the same laws that positive numbers obey is fundamental to the proof that $(-a)(-b)=ab$. This assumption is not likely to startle a ninth grader. It is a very natural assumption to make and because of its naturalness the postulational approach has a psychological advantage over the definitional approach. Furthermore it should be noted that the postulational approach avoids the appeal to authority used in most definitional approaches to $(-a)(-b)=ab$. Use of the postulational approach should make the pupil conscious of the role played by the "rules of computation" in the subject of algebra. Too often algebra is taught so as to minimize the pupil's appreciation of how fundamental these rules in algebra are. Not only does instruction often minimize the pupil's appreciation of the role played by the "rules of computation" in the subject of mathematics but methods of instruction often minimize the pupils' understanding of the rules themselves.

Since the treatment of x^0 and x^{-n} (n a positive integer) is so similar to the treatment of $(-a)(-b)=ab$, little space will be given to developing these two approaches to zero and negative exponents.

The Definitional Approach to $x^0=1$ and $x^{-n}=1/x^n$

The definitional approach, of course, simply defines $x^0=1$ and $x^{-n}=1/x^n$. However, elementary texts approach these definitions by first discussing the plausibility of making such a definition. This method is familiar to every high school teacher and need not be repeated here.

The Postulational Approach to $x^0=1$ and $x^{-n}=1/x^n$

The postulational approach again makes the basic assumption that the negative integral and zero exponents must obey all the laws of exponents developed for positive integral exponents. With this postulate it is possible to prove[2] that $x^0=1$. Thus

THEOREM. $x^0=1$.

Proof: $x^n \cdot x^m = x^{n+m}$.

Let $n=0$. The basic assumption permits this.

Then $x^0 \cdot x^m = x^{0+m} = x^m$.

Therefore x^0 must be 1 because this is the only value which multiplied by x^m will yield x^m as a result.

And again

THEOREM. $x^{-n}=1/x^n$.

Proof: $x^n \cdot x^m = x^{n+m}$.

Let $n=-n$. The basic assumption permits this.

$$x^n \cdot x^{-n} = x^{n-n} = x^0 = 1.$$

Hence the product of x^n and x^{-n} equals 1. Therefore x^n must be the reciprocal of x^{-n}. Or $x^{-n}=1/x^n$.

Proceeding in the same way it is possible to introduce all the fractional exponents used in elementary algebra courses.

What are some arguments for the postulational approach to $(-a)(-b)$, x^0 and x^{-n}? As has been said, the definitional approach if rigidly adhered to proves to be too difficult. This is due to mathematical immaturity of the ninth grade pupil. At this age pupils are not ready to grasp the significance of the use of such mathematical definitions as $x^0=1$. In order to offset this disadvantage teachers and authors have resorted to the use of illustrations to prepare the pupil psychologically for the definition to follow. The result is that the

[2] See Davis, H. T. "College Algebra," Prentice-Hall, Inc., New York, 1946, p. 52.

pupils, and sometimes the teachers as well, conceive of these illustrations as furnishing proof of what is actually a definition. Teachers and pupils miss the purpose of the illustration, namely, to show that the definitions have been wisely chosen in so far as the illustrations go. However the most telling criticism of the definitional approach (as used in elementary texts) is that it hides an assumption about operations with negative integers. This is hardly justifiable under any circumstances particularly when so many values are attainable when the assumption is brought out into broad daylight.

The postulational approach, it is contended, makes use of a perfectly natural assumption, namely, that the new numbers must obey all the laws previously agreed upon for positive numbers. In spite of its mathematical boldness, ninth grade pupils are willing to grant this assumption without argument. Once the assumption is granted the die is cast. It then follows that $(-a)(-b)=ab$ and $x^0=1$. Furthermore, the postulational approach makes use of simple proofs in the field of algebra. Proof has been traditionally neglected in algebra.[3] Some use of proof in algebra might be useful in preparing the student for plane geometry. Mathematically the use of proof in algebra is desirable because it broadens the student's concept of how and where proof is used. The pupil has the opportunity to become acquainted with the fact that proof as such is important in the study of algebra as well as in the study of geometry.

[3] In this connection see: "Mathematics in General Education," D. Appleton Century, New York, 1938, p. 201.

Attention National Council Members!

Because of the increased cost of printing THE MATHEMATICS TEACHER, the subscription price of the magazine (i.e. annual membership dues) were raised by the Board of Directors of the National Council of Teachers of Mathematics at their recent Annual Meeting at Atlantic City to $3.00 per year; single copies 40¢ postpaid. This increase in price takes effect October 1, 1947. This means that all members whose subscriptions expire in May should send $3.00 for their renewals and of course all new members should remit $3.00 also if they begin on or after October 1, 1947.

We are also compelled to increase the price of the 15th and 16th yearbooks to $2.00 each postpaid, effective May 1, 1947. We are sorry to have to increase our prices, but we hope that our members will all secure increases in salary for the coming year, so that our increases will not be burdensome. In the days ahead it is extremely important for every teacher of mathematics in the elementary and secondary schools to show his allegiance to The National Council by keeping up his membership in that organization. We know you will not fail us.—EDITOR

An Inquiry—What's Going on in Your School?

Please copy, fill out this page and mail today

The Commission on Post-War Plans of the National Council of Teachers of Mathematics would like to get a picture of what is happening in mathematics in typical schools—large and small. You can help by providing the information requested for your building or better still, for your whole organization if you are part of a city system. A report of trends will, in due time, be published in THE MATHEMATICS TEACHER. Feel free to be altogether frank—the report will in no way refer to you or your school without getting your special permission. Part III of this Inquiry will be published in the May number of THE MATHEMATICS TEACHER. Watch for it.

Part II

Name of School _____
Location of School _____
Name of person reporting _____
Address _____

1. Please fill in the following table:

Grades	Pupils in Grade	Pupils enrolled in a mathematics class
7		
8		
9		
10		
11		
12		

2. Check the factors that cut down your efficiency as a mathematics teacher.
   a. Large classes (state average size)
   b. Too many classes per week (state recitations per week)
   c. Lack of equipment (rulers, compasses, protractors, bulletin board, etc.)
   d. Lack of supplementary materials
   e. Outdated textbooks
   f. Required to teach unrelated courses
   g. Too many preparations
   h. No classroom that I can call my own
   i. Extracurricular duties
   j. Have to work at an out of school job to supplement income
   k. Inadequate training for my job
   l. Other teachers in department without proper training.
   m. Worry about moderate income
   n. lack of the right kind of supervision
3. As regards your present efficiency, and what you might do in an ideal situation, what percent do you think you are efficient? _____
4. Considering your total program in mathematics in relation to your whole school, what do you think is its efficiency rating in per cent? _____
5. Select your 10 best students, interview them and report:
   a. The number who are planning to be teachers of mathematics in high school
   b. For each pupil the main reason for his choice in *a*
   c. The number who intend to be teachers in a college or a university
   d. For each pupil, the main reason for his choice in *c*
   e. Research worker in government or industry
   f. For each pupil, the main reason for his choice in *e*.
6. How good a job would say most high school mathematics classes in your school are doing in providing young people with an intelligent basis for choosing mathematics as a profession—excellent, good, only fair, or poor?
7. How about giving your young people the training they need for successful college work in mathematics. Are most of the classes in the traditional sequential courses doing an excellent, good, only fair, or poor job there?
8. And how about giving young people an understanding of the role of mathematics in promoting general human welfare. The Second Report of the Commission includes a brief list that all pupils should learn if these things can be taught to them. Are your mathematics classes on the "second track" (general mathematics) in the effort to teach the mathematical know-how for better living doing an excellent, good, only fair, or poor job?

The report of Part I will be published in the October number of THE MATHEMATICS TEACHER.

Feel free to supplement your responses to the preceding inquiry by further comments on separate pages.

When you have filled out the blanks, which we hope will not take more than 10–15 minutes, please tear out this page and mail to Raleigh Schorling, Chairman, Commission on Post-War Plans, University High School, Ann Arbor, Michigan.

◆ IN OTHER PERIODICALS ◆

By NATHAN LAZAR

Midwood High School, Brooklyn 10, New York

The American Mathematical Monthly
February 1947, Vol. 54, No. 2

1. Price, G. B., "Some Identities in the Theory of Determinants," pp. 75–90.
2. Nielsen, K. L., "Industrial Experience for Mathematics Professors," pp. 91–96.
3. Mathematical Notes, pp. 96–100.
   (a) Klee, V. L., Jr., "On Completing a Determinant."
   (b) Parker, S. T., "A Pair of Recursion Relations."
4. Classroom Notes, pp. 100–106.
   (a) Johns, A. E., "The Reduced Equation of the General Conic."
   (b) Bird, M. T., "Simplification of Equations of Conics."
   (c) Wagner, R. W., "An Application of the Remainder Theorem."
5. Elementary Problems and Solutions, pp. 107–111.
6. Advanced Problems and Solutions, pp. 112–117.

The Australian Mathematics Teacher
November 1946, Vol. 2, No. 3

1. Witheford, C. E., "Formulae for the Triangles of Appolonius," pp. 49–53.
2. McCarthy, J. P., "Orthogonal Projection Methods in Trigonometry," pp. 54–58.
3. Meldrum, H. J., "Approximation," pp. 59–61.
4. Brother Liguori, "A Sequence in Trigonometry for non-Specialists," pp. 62–64.
5. Mathematical Notes, pp. 65–70.
   38. Wotton, F., "Proofs of $\sin(A \pm B)$ and $\cos(A \pm B)$."
   39. Nay, I. G., "Geometrical Proofs of $\tan(A \pm B)$."
   40. Schrumm, A. E., "The Solution of a Quadratic by Completing the Square."
   41. Gillings, R. J., "An Experiment in Graphs."
   42. Jones, Margaret, "An Interesting Proof of a Tricky Converse."
   43. McGann, Earl, "On Note 26"
   44. Clarke, J. H. C., "Pythagorean Integers."
   45. Cameron, W. T., "Pythagorean Integers."
   46. Carslaw, H. S., "Some References on Logarithmic and Exponential Functions."
   47. Brother Liguori, "Use of a Symbol for 'is similar to'."
   48. McGann, Earl, "On Note 20."

Bulletin of the Kansas Association of Teachers of Mathematics
October 1946, Vol. 21, No. 1.

1. Marm, Anna, "Survey of Freshman College Mathematics Courses in Kansas," p. 3.
2. Janes, W. C., "On Piling Oranges," pp. 4–5.
3. Bell, Kate, "Measurement," pp. 5–7.
4. Holroyd, Ina E., "Mathematical Symmetry in Nature," pp. 8–11.
5. Smith, Margaret Virginia, "Pleasure Derived from Word Problems," pp. 11–14.
6. Essentials for Functional Competence in Mathematics," pp. 14–15.

The Mathematical Gazette.
October 1946, Vol. 30, No. 291

1. "Euclid," p. 185.
2. Broadstreet, T. A. A., "*The Mathematical Gazette:* Our History and Aims," pp. 186–194.
3. Maunsell, F. G., "Why Does a Bicycle Keep Upright?" pp. 195–199.
4. Picken, D. K., "Sign, and Elementary Vector Ideas, in Plane Analytical Geometry and Trigonometry," pp. 200–208.
5. Robinson, R. T., "Spheres and Coinicoids through Twenty-Four Points," pp. 209–221.
6. Thébault, Victor, "Sur La Géométrie Récente du Tétraèdre," pp. 222–227.
7. Mathematical Notes, pp. 228–237.
   1928. F. W. K., "Distribution of Curriculum."
   1929. C. J. E., "On Certain Functional Equations and their Solutions."
   1930. "Addenda to Note 1845," (Find the Penny).
   1931. Dyson, F. J., "The Problem of the Pennies."
   1932. Ferguson, D. F., "The Solution of Triangles."
   1933. Mallison, H. V., "A Geometrical Query."
   1934. Paradine, C. G., "A Nomogram for the Solution of Right-angled Triangles."
   1935. Mason, J. I., "On the Angles of a Polygon."
8. Reviews, pp. 239–248.
9. Gleanings Far and Near (1495–1503), pp. 208–248.

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1. Program of the Silver Anniversary of the National Council of Teachers of Mathematics, pp. 4–9.
2. Messner, Madeline, "Presidential Message," p. 10.
3. Lumb, Leonard, "A Problem to Solve" (Editorial), p. 11.
4. Mallory, Virgil S., "Training Leaders in Mathematics," pp. 11–12.
5. Davis, David R., "Academic Life at Shrivenham American University," pp. 13–16.

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1. Hendricks, B. Clifford, "Why not Abolish Tests?" pp. 114–116.
2. Davis, H. T., "The Growth of Power and Its Economic Significance," pp. 124–138.
3. Bacon, Ralph Hoyt, "Integral Solutions of $x^2+y^2+z^2=r^2$," pp. 155–164.
4. Yudowitch, Kenneth L., "Putting the 'Simple' in Simple Harmonic Motion," pp. 183–185.
5. Nyberg, Joseph A., "Notes from a Mathematics Classroom," (continued), pp. 191–194.
6. Problem Department, pp. 195–199.
7. Books and Pamphlets Received, pp. 199–201.
8. Book Reviews, pp. 201–202.

BOOK REVIEW

Mathematics of Finance. By John A. Northcott. Rinehart and Company, N. Y., 1946. vi + 252 pages. Price, $3.00.

This text devotes 140 pages to a treatment of the most commonly met phases of the subject and 105 pages to the usual annuity and compound amount tables, a five place table of logs, and the theoretical tables of "amounts of 1 at compound interest for fractional periods." The remaining pages are devoted to answers to exercises and an index. The make-up is attractive, the print large, the annuity and compound interest tables give eight decimal places.

The scope of the subject matter is somewhat limited; series, logarithms, simple interest, compound interest, annuities, bonds, sinking funds and amortization, and depreciation, supply enough material for a good introductory course but leave the student without a knowledge of some phases of finance which are often encountered in practice, as well as some which are needed only occasionally. For example, the treatment of simple interest does not include bank discount, which is of common occurrence, nor are such topics as obsolescence and replacement of capital goods, building and loan calculations, industrial loans, the valuation of depleting assets, or good will valuation given consideration. It seems fair to say that the book covers the most important topics, those which are fundamental in finance, but should be classed as an introductory or elementary text on the subject.

In his presentation of the topics, the author shows that he is a good mathematician, and thinks logically and clearly. The treatment evidently assumes a knowledge of elementary algebra on the part of the learner. Such may be the case in many schools of business, but certainly not in all, in these days of graduation from high school without any mathematics being required. For example, on p. 13, it is hard to see how a student without a training in algebra could follow the argument, conducted in letters or algebraic symbols only. Would such students understand or obtain any real knowledge from the statement (p. 12),

"That is, if

$$a^x = N$$
$$x = \log_a N."$$

Most students of finance find it easier to follow a development, if it is first presented in the numbers of arithmetic, and then generalized into symbols of algebra, or formulas.

Here it must be stated that the author uses the labor saving device of avoiding the derivation of a different formula for each case that arises, when the problems can be solved by using the laws of geometric progression. And his explanations are clear and logical to those who can think in symbols.

As to the problems given in the Exercises after each new development of theory, they are numerous enough to give the required practice to give mastery of the theory, and many of them require real thinking and considerable ingenuity on the part of the student, if he is to obtain a solution.—W. S. SCHLAUCH.

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS MEMBERSHIP BY STATES—JANUARY 1947

Alabama	79	Maryland	112	Pennsylvania	412
Arizona	17	Massachusetts	229	Rhode Island	37
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Colorado	80	Mississippi	34	Tennessee	88
Connecticut	110	Missouri	133	Texas	200
Delaware	23	Montana	20	Utah	13
D. C.	105	Nebraska	79	Vermont	16
Florida	102	Nevada	4	Virginia	114
Georgia	40	New Hampshire	21	Washington	76
Idaho	4	New Jersey	277	West Virginia	43
Illinois	475	New Mexico	18	Wisconsin	179
Indiana	137	New York	513	Wyoming	19
Iowa	129	North Carolina	103	Canada	75
Kansas	166	North Dakota	19	U. S. Possessions	46
Kentucky	69	Ohio	348	Foreign	76
Louisiana	80	Oklahoma	97	Total	5928
Maine	34	Oregon	77		

MATHEMATICS

An Ode to the Queen of the Sciences

By D. B. Lloyd

Calvin Coolidge High School, Washington, D. C.

Queen Mathematics holds the key
To nature's laws o'er land and sea.
The rate of growth of organisms
Describes the curve of logarithms,
For log x equals t is all
We need say mathematical,
Expressing in the simplest form
Equations to which all conform.

Marine life hastens to comply,
The algae and the smaller fry.
The germs that plague us night and day
Can multiply no other way.
Amoebae join the host of things,
Excluding presidents and kings,
That follow mathematics' laws
Which makes effect to follow cause.

The atoms form by certain rules,
In crystals, blocks, and molecules;
The pattern destines what they are,
As sugar, diamond, coal or tar.
Though broken up exceeding fine
Their atoms still remain in line,
The polyhedral forms they choose
Yield properties they never lose.

The sea-shells on life's sandy shore
Are shiny shells and nothing more
To him who doesn't wish to know
Nor care to ponder how they grow.
The fish and fowl that swim and fly
Are naval architects' delight;
Their streamlined curves both please the eye
And aid in movement and in flight.

With spiral curves the conch-shell grows
A flaring base and pointed nose,
Can give his shell an angle turn
And disappear from stem to stern,
The helix of his armor-plate
Increases speed to higher rate;
The power of such a driven screw
A stern-wheel paddle never knew.

To understand Dame Nature's curves
And pay her homage she deserves
We need to seek Queen Mathematics
And scrutinize her acrobatics.
Her laws of reason hold the key
To things that are and things to be;
From out of chaos order rises
When mathematics analyzes.

That very law that molds a tear
And bids it trickle from its source,
That same law holds the earth a sphere
And keeps the planets in their course.
From atoms small to stars on high,
Throughout the length of time and sizes
In universes far and nigh,
Forever God geometrizes.

The moral to this lesson true
I wish to leave with each of you.
Now listen close,—it is but this:
Hold not that ignorance is bliss!
'Tis folly to be wise, you've heard—
Was never spoke by that wise bird
And knows the why's of which I tell.
Who learns his mathematics well

KENISTON TULLY

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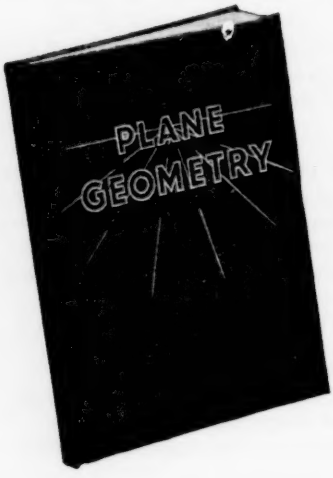

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